A Uniform Equivariant Model for Cubical Type Theory

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1 Introduction

In recent years there has been a rise in interest in homotopy type theory, a form of dependent type theory that treats identities as homotopies between the relevant objects. Notably there is an axiom that establishes an equivalence between homotopy equivalences of two types and identities between these types. The model structure that was built by Voevodsky to prove consistency of these axioms, is built on simplicial sets, and the well known Kan-Quillen model structure[KL21]. One might even say that this is not only a homotopical version of type theory, but the language of homotopy types. This aligns very well with interpretation, types are interpreted as fibrations. But from a constructive view point this was not satisfying as parts of the interpretation were based on classical logic. A bit later it was shown that this model inherently needs some classical principles [BCP15].

One of the approaches to get a constructive model was to go over to cubical models, the first of these is due to Bezem, Coquand, and Huber [BCH14]. From there different variants of cubical models where considered [CCHM; AHH18; CHM18] one of them the Cartesian cubes [Awo18]. These are all different model construction with slightly different attached type theories. There happened a lot of work to unify some of these constructions [Ang+21].

It is not at all clear that these give rise to model structures, but also not very surprising considering the languages from which these are build try to capture homotopical notions. Now one might ask if these induced homotopy theories are again equivalent to the original Kan-Quillen model structure. For this we have a multiple negative results. These results are not well documented and the reference I could find is on this mailing list [Mail]. For the model structure induced on the Dedekind cubes this question remains an open. The equivariant model structure on Cartesian cubes discussed in this thesis is the first construction for which this equivalence is known. A little later Cavallo and Sattler found that the in the case with only one connection the equivalence to spaces can also be shown [CS22].

This thesis gives a proof that the uniform equivariant model of cubical type theory is equivalent to spaces. This model came out of the joint work of Awodey, Cavallo, Coquand, Riehl, and Sattler. The thesis is split in three parts, the first part are short introductions to topics of category theory that one might not be familiar with. The reader acquainted with these might safely skip them. The second part introduces the main object, the topos $\widehat{\square}$, and the equivariant premodel structure on it. It then continues by constructing universes for those fibrations, and uses them to prove that the premodel structure is indeed a model structure. The third part shows the equivalence of this model structure to spaces.

In section two we first describe the site and afterwards the equivariant premodel structure. We spend then most of the section with the construction of fibrant universes. This mostly follows [Awo23a]. The main idea of this argument is to internalize the structure of the algebraic factorization system, attached to the fibrations. These are called classifying types. These can then be used to extract universes from Hofmann-Streicher universes.

In section three we show equivalence to spaces and follow the argument sketched in [Rie20]. To compare the simplex category Δ to the Cartesian cubes $\square$, we pass through the Dedekind cubes. We establish a pair of left Quillen functors that induce an equivalence of the desired homotopy categories. An important fragment of the proof was laid out in [Sat19].

When this project started there was no written account on this model, and I tried to remedy this fact. So the content is mainly based on a talk given by Riehl [Rie20], the account about premodel structures in [CS22, Section 3], and universe construction techniques from [Awo23a]. Shortly before this project was finished, an extensive written report on this subject was released by the original creators of this model structure [ACCRS]. I changed some formulations in the universe construction to highlight the similarities in the argument. For example there are now explicit mentions of cubical species in the argument. We also verify an older argument for the equivalence to spaces that is no longer present in [ACCRS] in this form.

2 Preliminaries

2.1 Algebraic Weak Factorization Systems

In this section, we will revisit the basic ideas of algebraic weak factorization systems (awfs). We won't use them much explicitly through this paper, but we need one major result about them. Also, while we don't talk about them explicitly, their ideas permeate through most of the arguments. We will only repeat the most basic definitions and ideas, which will be enough to understand this document. For a much more complete and in depth discussion, see [Rie11; BG16a; BG16b]. This introduction follows the approach of [Rie11, Section 2]. If the reader is already familiar with this concept, they might safely skip this section.

We start this section with some observations about regular functorial weak factorization systems (wfs). For the remainder of this section we write $\mathrm{E} : \mathcal{A}^{\rightarrow} \to \mathcal{A}^{3}$ as the factorization functor of some functorial wfs $(\mathcal{L}, \mathcal{R})$. We are going to write $\mathbf{d}^0, \mathbf{d}^2 : \mathcal{A}^3 \to A^{\rightarrow}$ for the functors induced by $d_0, d_2 : \mathbb{2} \to \mathbb{3}$. L and R for the endofunctors $\mathcal{A}^{\to} \to \mathcal{A}^{\to}$ that are given by $\mathbf{d}^2\mathrm{E}$ and $\mathbf{d}^0\mathrm{E}$, the projection to the left or right factor of the factorization. For a given $f : X \to Y$, we call the factoring object E_f.

$$\begin{tikzcd} & \mathrm{E}_f \arrow[dr, "\mathrm{R}(f)"] & \\ X \arrow[ur, "\mathrm{L}(f)"] \arrow[rr, "f"] & & Y \end{tikzcd}$$

For now, we are interested in witnessing if some map is in the right class (or dually left class). Or in other words, attaching some kind of data to a right map from which we could deduce all solutions of the required lifting problem. This is indeed possible. Assume that f is a right map, then a retraction r_f of $\mathrm{L}(f)$ would suffice. Assume we had some left map f' and a lifting problem given by (g, h). We can then factor this with the help of E.

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "\mathrm{L}(f')"] \arrow[dd, bend right=60, "f'"'] & X \arrow[d, "\mathrm{L}(f)"] \arrow[dd, bend left=60, "f"] \\ \mathrm{E}_{f'} \arrow[r, "\phi_{g,h}"] \arrow[d, "\mathrm{R}(f')"] & \mathrm{E}_f \arrow[u, bend left, "r_f"] \arrow[d, "\mathrm{R}(f)"] \\ Y' \arrow[r, "h"] \arrow[ur, dashed] & Y \end{tikzcd}$$

And then compose the solution for the whole lifting problem from the lifting of the problem $(gL(f), h)$ with r_f. That this is a solution is guaranteed by r_f being a retract $r_f L(f) = \mathsf{id}$. Dually we can witness f' being a left map by supplying a split $s_{f'}$ of $R(f')$. If we did both at the same time we automatically get a canonical choice of lifts.

$$\begin{tikzcd} X' \arrow[r, "g"] \arrow[d, "\mathrm{L}(f')"] \arrow[dd, bend right=60, "f'"'] & X \arrow[d, "\mathrm{L}(f)"] \arrow[dd, bend left=60, "f"] \\ \mathrm{E}_{f'} \arrow[r, "\phi_{g,h}"] \arrow[d, "\mathrm{R}(f')"'] & \mathrm{E}_f \arrow[u, bend left, "r_f"] \arrow[d, "\mathrm{R}(f)"'] \\ Y' \arrow[r, "h"] \arrow[u, bend right, "s_{f'}"'] & Y \end{tikzcd} \tag{1}$$

Namely for a lifting problem (g, h) the map $r_f \phi_{g,h} s_{f'}$, and if we make r_f and $s_{f'}$ part of the data of a right- (left-) map the chosen lifts are even functorial. The next question one might rightfully ask, if we can always find such a witness. And the answer is happily yes. We just need to make up the right lifting problem.

$$\begin{tikzcd} X' \arrow[r, "\mathrm{L}(f')"] \arrow[d, "f'"'] & \mathrm{E}_{f'} \arrow[d, "\mathrm{R}(f')"] \\ Y' \arrow[r, equal] \arrow[ur, dashed, "s_{f'}"] & Y' \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, equal] \arrow[d, "\mathrm{L}(f)"'] & X \arrow[d, "f"] \\ \mathrm{E}_f \arrow[r, "\mathrm{R}(f)"'] \arrow[ur, dashed, "r_f"] & Y \end{tikzcd}$$

We can also repack this information in a slightly different way, f is a right map exactly if f is a retract of $\mathcal{R}(f)$ in $\mathcal{A}/_Y$. And f' is a left map precisely if f' is a retract of $\mathrm{L}(f)$ in $X'/\mathcal{A}$.

$$\begin{tikzcd} X \arrow[r, "\mathrm{L}(f)"] \arrow[d, "f"] & E_f \arrow[r, dashed, "r_f"] \arrow[d, "\mathrm{R}(f)"] & X \arrow[d, "f"] \\ Y \arrow[r, equal] & Y \arrow[r, equal] & Y \end{tikzcd} \qquad \begin{tikzcd} X' \arrow[r, equal] \arrow[d, "f'"] & X' \arrow[r, equal] \arrow[d, "\mathrm{L}(f)"] & X' \arrow[d, "f'"] \\ Y' \arrow[r, dashed, "s_{f'}"] & E_{f'} \arrow[r, "\mathrm{R}(f')"] & Y' \end{tikzcd}$$

If we focus on the diagram at the left-hand side, we can also see it as a morphism $\eta_f : f \to \mathrm{R}(f)$ in $\mathcal{A}^{\to}$, completely dictated by E and thus natural in f, and a morphism $\alpha : \mathrm{R}(f) \to f$, such that $\alpha\eta_f = \mathsf{id}$. If we reformulate what we have just observed, we get to the following.

2.1.1 Observation. In a functorial wfs $(\mathcal{L}, \mathcal{R})$ on $\mathcal{A}$, $\mathrm{L} : \mathcal{A} \to \mathcal{A}$ is a copointed endofunctor and R is pointed endofunctor, where the counit $\varepsilon : \mathrm{L} \to \mathsf{Id}$ is given by the squares $\varepsilon_f := \begin{tikzcd} \cdot \arrow[r, equal] \arrow[d, "\mathrm{L}(f)"] & \cdot \arrow[d, "f"] \\ \cdot \arrow[r, "\mathrm{R}(f)"] & \cdot \end{tikzcd}$ and the unit $\eta : \mathsf{Id} \to \mathrm{R}$ by $\eta_f := \begin{tikzcd} \cdot \arrow[r, "\mathrm{L}(f)"'] \arrow[d, "f"] & \cdot \arrow[d, "\mathrm{R}(f)"'] \\ \cdot \arrow[r, equal] & \cdot \end{tikzcd}$. $\mathcal{L}$ is precisely the class of L-coalgebras and $\mathcal{R}$ the class of R-algebras.

One should think of these (co)algebras as morphism with a choice of liftings. At this point, we might try to get rid of the wfs $(\mathcal{L}, \mathcal{R})$ as input data and might to try to recover it from the factorization functor. And that works described by the methods above, but only if we know that this factorization functor comes from a wfs. If we just start with an arbitrary factorization functor, we still get that all R-algebras right lift to all L-algebras and vice versa, but in general $\mathrm{R}(f)$ will not be a R-algebra. One way to solve this problem is by adding a second natural transformation $\mathrm{RR} \to \mathrm{R}$, such that the neccessary data commute, making R a monad (and dually L a comand).

2.1.2 Definition. An *algebraic weak factorization system (awfs)* is given by a functor $\mathrm{E} : \mathcal{A}^{\rightarrow} \rightarrow \mathcal{A}^{3}$ and two natural transformations δ and μ. We write $(\mathrm{L}, \varepsilon)$ where $\mathrm{L} \coloneqq \mathbf{d}^{2}\mathrm{E}$ and (R, η) where $\mathrm{R} \coloneqq \mathbf{d}^{0}\mathrm{E}$ for the two induced pointed endofunctors $\mathcal{A}^{\rightarrow} \rightarrow \mathcal{A}^{\rightarrow}$. We require that $(\mathrm{L}, \varepsilon, \delta)$ is a comonad and (R, η, μ) is a monad.[1]

2.1.3 Notation. If the rest of the data is clear from context we will only specify the comonad and monad and say (L, R) to be an algebraic weak factorization system.

2.1.4 Remark. Neither the L-coalgebras nor the R-algebras are in general closed under retracts, if we think of them as comonad and monad. But we get a full wfs by taking the L-coalgebras and R-algebras if we only regard them as the pointed endofunctor, which is the same as the retract closure of the algebras in the (co)monad sense.

In light of this remark we can pass back from any awfs to a wfs. We can think of this operation as forgetting the choice of liftings.

2.1.5 Definition. Let L, R be an awfs. We will call $(\mathcal{L}, \mathcal{R})$ the underlying wfs of (L, R), where $\mathcal{L}$ is the class of L-coalgebras (as a pointed endofunctor) and $\mathcal{R}$ is the class of R-algebras (as a pointed endofunctor).

2.1.6 Remark. Dropping the algebraic structure is not a lossless operation. Even though the (co)pointed endofunctors and (co)algebras with respect to those endofunctors can be recovered (with enough classical principles), the unit and counit might not have been unique. And thus also not the category of (co)algebras regarding the (co)monad structure.

We will end this discussion with a few definitions and a theorem that we will later need. While we think the reader is now well prepared to understand the statements and their usefulness, we are aware that we didn't cover enough theory to understand its inner workings.

2.1.7 Definition. Let $\mathcal{J}$ be category and $J : \mathcal{J} \rightarrow \mathcal{A}^{\rightarrow}$ be a functor. Then an object of the category $J^{\boxslash}$ of *right J-maps* is a pair (f, j) with f in $\mathcal{A}^{\rightarrow}$ and j a function

[1]Most modern definitions require additionaly that a certain induced natural transformation to be a distributive law of the comonad over the monad. While we recognise its technical important,but we feel that it is distracting from our goal to get the general ideas across.

that assigns for every object i in $\mathcal{J}$, and for every lifting problem

$$\begin{tikzcd}
L \arrow[r, "\alpha"] \arrow[d, "J(i)"'] & X \arrow[d, "f"] \\
M \arrow[r, "\beta"] \arrow[ur, dashed, "j(i,\alpha,\beta)"] & Y
\end{tikzcd}$$

a specified lift $j(i, \alpha, \beta)$, such that for every $(a, b) : k \to i$ in $\mathcal{J}$, the diagram

$$\begin{tikzcd}
L' \arrow[r, "a"] \arrow[d, "J(k)"'] & L \arrow[r, "\alpha"] \arrow[d, "J(i)"] & X \arrow[d, "f"] \\
M' \arrow[r, "b"] \arrow[urr, dashed, "j(i,\alpha a,\beta b)"] & M \arrow[r, "\beta"] \arrow[ur, dashed, "j(i,\alpha,\beta)"'] & Y
\end{tikzcd}$$

commutes. And morphisms are morphisms in $\mathcal{A}^{\to}$ that preserve these liftings.

2.1.8 Remark. This is even a functor $(-)^{\boxslash} : \mathbf{Cat}/\mathcal{A}^{\to} \to \left(\mathbf{Cat}/\mathcal{A}^{\to}\right)^{\mathrm{op}}$

2.1.9 Remark. There is an adjoint notion of left lifting.

2.1.10 Remark. This is a strong generalization from the usual case, where one talks about sets (or classes) that lift against each other. If one believes in strong enough choice principles, then the usual case is equivalent to $\mathcal{J}$ beeing a discrete category and J some monic functor.

We will now turn to a theorem that will provide us with awfs that are right lifting to some functor J. It is (for obvious reasons) known as Garners small object argument.

2.1.11 Theorem (Garner [Gar09; Rie11, Theorem 2.28, Lemma 2.30]). Let $\mathcal{A}$ be a cocomplete category satisfying either of the following conditions.

- $(*)$ Every $X \in \mathcal{A}$ is α_X-presentable for some regular cardinal α_X.
- $(\dagger)$ Every $X \in \mathcal{A}$ is α_X-bounded with respect to some proper, well-copowered orthogonal factorization system on $\mathcal{A}$, for some regular cardinal α_X.

Let $J : \mathcal{J} \to \mathcal{A}^{\to}$ be a category over $\mathcal{A}^{\to}$, with $\mathcal{J}$ small. Then the free awfs on $\mathcal{J}$ exists, its category of R algebras is isomorphic to $J^{\boxslash}$, and the category of R-algebras is retract closed.

2.2 Leibniz Construction

We will use a well-know construction in homotopy theory to elegantly construct a lot of interesting objects, the Leibniz construction. This section will mostly give a definition and some examples to get familiar with this construction. If the reader is already familiar with it, they might skip this section without any problems. We start by giving the definition.

2.2.1 Definition (Leibniz Construction). Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be categories and $\mathcal{C}$ have finite pushouts. Let $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a bifunctor. Then we define the *Leibniz Construction* $\widehat{\otimes} : \mathcal{A}^{\to} \times \mathcal{B}^{\to} \to \mathcal{C}^{\to}$ to be the functor that sends $f : A \to A'$ in $\mathcal{A}$ and $g : B \to B'$ in $\mathcal{B}$, to $f \mathbin{\widehat{\otimes}} g$ as defined via the following diagram.

$$
\begin{tikzcd}
A \otimes B \arrow[r, "f\otimes \mathrm{id}"] \arrow[d, "\mathrm{id}\otimes g"] & A' \otimes B \arrow[d] \arrow[ddr, bend left, "\mathrm{id}\otimes g"] & \\
A \otimes B' \arrow[r] \arrow[drr, bend right, "f\otimes \mathrm{id}"'] & A \otimes B' \coprod_{A\otimes B} A' \otimes B \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "f \mathbin{\widehat{\otimes}} g"] & \\
 & & A' \otimes B'
\end{tikzcd}
$$

If $\otimes$ is the tensor functor of a monoidal category, we also call it the *Leibniz product* or *pushout-product*. If $\otimes$ is the functor application functor, we also call it the *Leibniz application*.

The following examples are true for any category of nice spaces. We will state them for simplicial sets as the reader is probably familiar with them.

2.2.2 Example. Let $\delta_k : * \to \Delta^1$ for $k \in \{0, 1\}$ be one of the endpoint inclusions into the interval and $i : \partial\Delta^n \to \Delta^n$ the boundary inclusion of the n-simplex. Then $\delta_k \mathbin{\widehat{\times}} i$ is the inclusion of a prism without its interior and one cap into the prism.

This gives us a description of a possible set of generating trivial cofibrations. This will be an ongoing theme. We can even observe something stronger. If we would replace the boundary inclusion of the n-simplex with the boundary inclusion of an arbitrary space X. We get the inclusion of the cylinder object of X without a cap and its interior into the cylinder.

2.2.3 Observation. If f and g are cofibrations then $f \mathbin{\widehat{\times}} g$ is too. If f or g is a trivial cofibration, then so is $f \mathbin{\widehat{\times}} g$.

This has far reaching consequences and one can define the notion of a monoidal model category where this one of the axioms for the tensor functor. This axiom basically states that the tensor product behaves homotopicaly. For more detail see [Hov07]. We are not going to need much of this theory explicitly. But it is worthy noting that all examples of model categories that we are going to encounter are of this form.

We will also encounter basically the same construction as Example 2.2.2 in another way. We can get the Cylinder object functorialy in $\widehat{\Delta}$, such that the cap inclusions are natural transformations. The inclusion the Leibniz product produces the inclusion in a natural manner via the Leibniz application.

2.2.4 Example. Let the functor application functor $@ : \mathcal{C}^{\mathcal{C}} \times \mathcal{C} \to \mathcal{C}$ defined as $F@x := F(x)$. Let $\mathbb{I} \times (-)$ be the functor that sends X to its cylinder object (also known as the product with the interval). Let $\delta_k : \mathsf{Id} \to \mathbb{I} \otimes (-)$ be one of the, the natural transformation that embeds the space in one of the cylinder caps. Let $\partial X \to X$ be the boundary inclusion of X. Then $\delta_k \widehat{@}(\partial X \to X)$ is the filling of a cylinder with one base surface of shape X.

This kind of construction will later play an important role in the construction of our desired model categories. And as we will use constructions like this more often we add a bit of notation for it.

2.2.5 Notation. Let $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a bifunctor, A an object of $\mathcal{A}$, $F : \mathcal{B} \to \mathcal{C}$ a functor, and $\eta : F \Rightarrow A \otimes (-)$ a natural transformation. We write $\eta \mathbin{\widehat{\otimes}} (-) := (\eta \otimes (-))\widehat{@}(-)$.

Here is a property that we will exploit fairly often.

2.2.6 Proposition. Since every topos is adhesive [LS05], monomorphisms are stable under the pushout-product and the Leibniz application of $A \times (-)$ in every topos.

The Leibniz construction has a dual construction, the Leibniz pullback construction.

2.2.7 Definition (Leibniz pullback Construction). Let $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ be categories and $\mathcal{C}$ have finite pullbacks. Let $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a bifunctor. Then we define the *Leibniz pullback construction* $\check{\otimes} : \mathcal{A}^{\to} \times \mathcal{B}^{\to} \to \mathcal{C}^{\to}$ to be the functor that sends $f : A \to A'$ in $\mathcal{A}$ and $g : B \to B'$ in $\mathcal{B}$ to $f \check{\otimes} g$ as defined by the following diagram.

$$\begin{tikzcd}
A \otimes B \arrow[dr, dashed] \arrow[drr, bend left, "f\otimes\mathrm{id}"] \arrow[ddr, bend right, "\mathrm{id}\otimes g"'] & & \\
& A \otimes B' \prod_{A'\otimes B'} A' \otimes B \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & A' \otimes B \arrow[d, "\mathrm{id}\otimes g"] \\
& A \otimes B' \arrow[r, "f\otimes\mathrm{id}"] & A' \otimes B'
\end{tikzcd}$$

If $\otimes$ is the exponential functor, we also call it the *Leibniz pullback Hom*, *pullback-power* or *pullback-hom*. If $\otimes$ is the functor application functor we also call it the *Leibniz pullback application*.

Analog to the notation above we introduce the following notation

2.2.8 Notation. Let $\otimes : \mathcal{A} \times \mathcal{B} \to \mathcal{C}$ be a bifunctor, A be an object of $\mathcal{A}$, $F : \mathcal{B} \to \mathcal{C}$ be a functor , and let $\eta : F \Rightarrow A \otimes (-)$ be a natural transformation. We write $\eta \widehat{\otimes} (-) := (\eta \otimes (-)) \widehat{@} (-)$. As it is common in the literature, we also write the following for the special case $A \otimes (-) = (-)^A$, $\eta \Rightarrow (-) := \eta \check{\otimes} (-)$.

Dual to our observation above it will be a theme that the pullback-power sends fibrations to trivial fibrations. Or more formally:

2.2.9 Lemma ([RV14, Lemma 4.10])**.** There is an adjunction $\eta \mathbin{\hat{\times}} (-) \dashv \eta \Rightarrow (-)$ between the pushout-product and the pullback-power.

2.2.10 Remark. This holds in much more generality, for details see [ACCRS, Lemma 2.1.15].

2.2.11 Remark. Our main application of this lemma will be that it gives us the ability to transpose lifting problems between cofibrations and trivial fibrations, to lifting problems between trivial cofibrations and fibrations.

We will encounter another form of Leibniz application, which at first glance does not has a lot to do with homotopy. Though, one might read this as getting the component of a natural transformation some sort of a homotopic construction.

2.2.12 Lemma. Let $F, G : A \to B$ be functors, $\eta : F \Rightarrow G$ a natural transformation, and $\mathcal{A}$ have an initial object $\varnothing$. Let F and G preserve the initial object. We write $\varnothing \to X$ for the unique map. Then we get $\eta \mathbin{\widehat{@}} (\varnothing \to X) = \eta_X$.

Proof. We will draw the diagram from the definition

$$\begin{tikzcd}
F(\varnothing) \arrow[r, "F(\varnothing\to X)"] \arrow[d, "\eta_\varnothing"] & F(X) \arrow[d] \arrow[ddr, bend left, "\eta_X"] & \\
G(\varnothing) \arrow[r] \arrow[drr, bend right, "G(\varnothing\to X)"'] & G(\varnothing) \coprod_{F(\varnothing)} F(X) \arrow[ul, phantom, "\ulcorner", very near start] \arrow[dr, dashed, "\eta\hat{@}(\varnothing\to X)"] & \\
& & G(X)
\end{tikzcd}$$

If we substitute $F(\varnothing)$ and $G(\varnothing)$ with $\varnothing$ the claim directly follows. □

3 Cubical Sets

3.1 Category of Cubes

The first question we need to answer, what site to consider. In the simplicial world there is more or less a consensus, what exactly the site should be. In the cubical world there are multiple models with slightly different sites. Our main protagonist will be the Cartesian model, but we will also need to introduce the Dedekind model, as it will appear in an argument later. For a comparative analysis we direct the reader to [CMS16] and [Ang+21].

3.1.1 Cartesian cubes

The most uniform description is probably in form of Lawvere theories, but our site has a more direct description [Rie20].

3.1.1 Definition. Let FinSet be the category whose objects are the subsets of $\mathbb{N}$, of the form $\{0, .., n\}$, for some $n \in \mathbb{N}$, and as morphisms functions between them.

3.1.2 Remark. Up to equivalence this is the category of finite sets. Choosing a skeleton will circumvent later size issues, but for convenience we will pretend that these are finite sets and not always re-index our sets to have an initial segment of $\mathbb{N}$.

3.1.3 Definition. Let $\square \coloneqq \mathrm{FinSet}_{*}^{*}{}^{\mathrm{op}}$ be the opposite category of double-pointed finite sets.

An alternative definition would be

3.1.4 Definition. Let $\square$ be the Lawvere-theory of the type theory with a single type $\mathbb{I}$, and two constants $\bot : \mathbb{I}$ and $\top : \mathbb{I}$.

3.1.5 Remark. In light of Definition 3.1.3 and Definition 3.1.4 we sometimes talk about morphisms as functions $X + 2 \to Y + 2$ preserving both points in 2 or equivalently ordinary functions $X \to Y + 2$.

We will now spend some time exploring, why this might be viewed as the cube category. We write $\{\bot|a, b, c|\top\}$ for double pointed finite sets, where $\bot$ and $\top$ are the two distinguished points. A cube of dimension n is now interpreted as such a set with $n + 2$ (n without the distinguished points) many elements. We think of these as dimension variables. We could for example think about the zero dimensional cube (a point) $\{\bot|\top\}$ and the 2 dimensional cube (a square) $\{\bot|a, b|\top\}$. How many maps do we expect from the line, into the point? Exactly one. And this is also exactly what we get, a fitting map in $\mathbf{FinSet}_*^{*\ \mathrm{op}}$ corresponds to a function $\{\bot|\top\} \to \{\bot|a, b|\top\}$ preserving $\bot$ and $\top$, this is already unique. In the other direction we have four maps. These encode the corners. For example the corner "$(\bot, \bot)$" gets encode by sending a and b to $\bot$.

Let us increase the dimensions a bit and look at the maps from a 1 dimensional cube (a line segment) $\{\bot|x|\top\}\}$ into the square. These correspond to functions from $\{a, b\} \to \{\bot, x, \top\}$. We can think of this as a dimensional description of a line in a square. We have 2 dimensions in which the line can move a and b. And in these two dimensions the line can either be constant (sending this dimension to $\bot$ or $\top$, or can advance along its dimension (sending the dimension to x). So we get the degenerate corner points (as lines), by sending both a and b to $\bot$ or $\top$. We get the four sides of the square by sending one of a or b to $\bot$ or $\top$. The two sides that run along the dimension a send a to x. One function is left, sending both a and b to x. This is the line that advances in both dimensions. Or in other words the diagonal.

3.1.6 Remark. Our cube category is closed under taking products. This differs from the simplicial case. This will turn out to be very handy and important later.

As we will need some of these element throughout the document again we will give them names.

3.1.7 Definition (Notation for cubes)**.** We call:

- $[0] \coloneqq \{\bot|\top\}$
- $[1] \coloneqq \{\bot|x|\top\}$
- $\deg(x) \coloneqq |x| - 2$

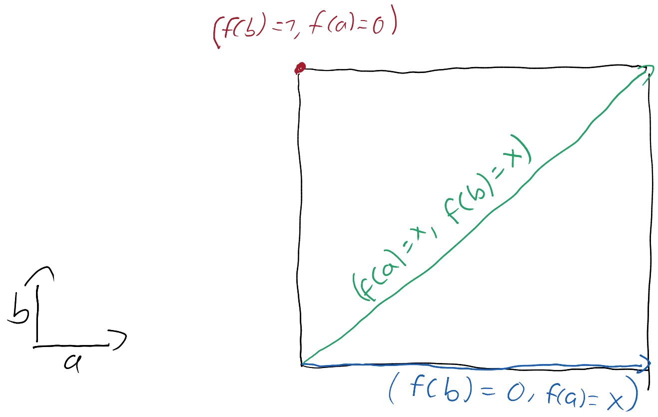

Figure 1: A square with some lines mapped into it

- $\mathrm{d}_k : [0] \to [1]$ with $k \in \{\top, \bot\}$, be the map given by the function that sends $x \mapsto k$

Also while $\square$ is not a strict Reedy-category (as Δ is) it almost is, more precisely it is an Eilenberg-Zilber category. Most of Reedy theory from the simplicial case can be salvaged. This will play an integral part in the proof that our model structure on $\widehat{\square}$ is equivalent to spaces.

3.1.8 Definition (generalized Reedy-category). A *generalized Reedy category* is a category $\mathcal{C}$ together with two wide subcategories $\mathcal{C}_+$ and $\mathcal{C}_-$ and a function $d : \mathrm{ob}(\mathcal{C}) \to \alpha$ called *degree*, for some ordinal α, such that

1. every non-isomorphism in $\mathcal{C}_+$ raises degree,
2. every non-isomorphism in $\mathcal{C}_-$ lowers degree,
3. every isomorphism in $\mathcal{C}$ preserves degree,
4. $\mathcal{C}_+ \cap \mathcal{C}_-$ is the core of $\mathcal{C}$
5. every morphism factors as a map in $\mathcal{C}_-$ followed by a map in $\mathcal{C}_+$, uniquely up to isomorphism
6. if $f \in \mathcal{C}_-$ and θ is an isomorphism such that $\theta f = f$, then $\theta = \mathrm{id}$

This will be an important concept later, but in our special case we can do even better

3.1.9 Definition (Eilenberg-Zilber category). An *Eilenberg-Zilber category* or short *EZ-category* is a small category $\mathcal{C}$ equipped with a function $d : \mathrm{ob}(\mathcal{C}) \to \mathbb{N}$ such that

1. For $f : x \to y$ a morphism of $\mathcal{C}$:
   a) If f is an isomorphism, then $d(x) = d(y)$.
   b) If f is a noninvertible monomorphism, then $d(x) < d(y)$.
   c) If f is a noninvertible split epimorphism, then $d(x) > d(y)$.
2. Every morphism factors as a split epimorphism followed by a monomorphism.
3. Every pair of split epimorphisms in $\mathcal{C}$ has an absolute pushout.

3.1.10 Remark. Clearly every EZ-category is a generalized Reedy category

3.1.11 Remark. A lot of homotopical construction one knows from Reedy categories can be carried out with EZ-categories. A word of caution: if we try to do the usual Reedy style decomposition of a presheaf into a cell-complex, such that the attached cells are boundary inclusions in the representables, we will fail. Instead we need to consider all quotients of boundary inclusions in quotients of representables. For more details see [RV14; Shu; Cam23].

3.1.12 Theorem. $\square$ together with deg is an EZ-category

Before we give a proof we will give a technical lemma about absolute pushouts of split epis.

3.1.13 Lemma. Let

$$\begin{tikzcd}
A \arrow[r, two heads, "b"'] \arrow[d, two heads, "c"] & B \arrow[l, dashed, bend right, "b'"'] \arrow[d, two heads, "d_1"'] \\
C \arrow[u, dashed, bend left, "c'"] \arrow[r, "d_2"] & D \arrow[u, dashed, bend right, "d_1'"']
\end{tikzcd}$$

be a diagram in a category $\mathcal{C}$ of split epis such that, $bc' = d_1' d_2$. Then it is already a pushout square.

Proof. We look at an arbitrary co-cone

$$\begin{tikzcd}
A \arrow[r, two heads, "b"] \arrow[d, two heads, "c"] & B \arrow[d, "e_1"] \\
C \arrow[r, "e_2"] & E
\end{tikzcd}$$

We can compose this square with (c', d_1') yielding

$$\begin{tikzcd}
C \arrow[r, "d_2"] \arrow[d, "c'"] \arrow[dd, bend right=60, "\mathrm{id}"'] & D \arrow[d, "d_1'"] \\
A \arrow[r, two heads, "b"] \arrow[d, two heads, "c"] & B \arrow[d, "e_1"] \\
C \arrow[r, "e_2"] & E
\end{tikzcd}$$

We get our candidate map from $e_1 d_1' : D \to E$. We need to show that it commutes with e_1 and d_1 as well as e_2 and d_2. The second case $e_1 d_1' d_2 = e_2$ is immediately obvious from the diagram. To check that $e_1 d_1' d_1 = e_1$ it is enough to check that $b e_1 d_1' d_1 = b e_1$, as b is epic. Which is a straight forward diagram chase, if one adds all the not drawn arrows to the diagram above. Since d_1 are epic, $e_1 d_1'$ is unique in fulfilling already one of the two conditions. $\square$

3.1.14 Remark. As we see in the proof, we don't need b to be split, but our application needs the split, as split epis are preserved by pushouts and general epis are not.

3.1.15 Remark. As split epis and commutativity is preserved by all functors, pushouts of this form are absolute pushouts.

As we will use the following statement multiple times in the proof we put it into its own lemma.

3.1.16 Lemma. Every epi in $\square$ splits. Or dually every mono in $\mathbf{FinSet}^*_*$ has a retraction. This can be constructed in a way that for a mono f in $\mathbf{FinSet}^*_*$ the corresponding contraction sends everything not in the image of f to $\perp$.

Proof. Let f be an epi in $\square$. It is thus a monomorphism in $\mathbf{FinSet}^*_*$. We construct a retraction g that will be a split in $\square$. Let $g(x)$ by the unique pre-image of f if it is not empty. If it is empty we choose any element (for example $\perp$). It is trivial to verify that this is a retraction. As f needs to preserve the distinguished points, g does too. $\square$

Proof of Theorem 3.1.12. Property 1 follows from basic facts about surjectivity and injectivity between finite sets. Property 2 needs us to find a (epi,mono) factorization in $\mathbf{FinSet}^*_*$. This factorization can be obtained in the same way one would obtain it **Set**, by factoring an morphism through its image. It is straight forward to verify, that the distinguished points are preserved by those maps. And the fact that the epi (in $\square$) splits follows from Lemma 3.1.16.

For property 3, we only need find a pushout and splits of the form described in Lemma 3.1.13. So let $c : A \to C$ and $b : A \to B$ split epis. We will do this by constructing the dual diagram in $\mathbf{FinSet}^*_*$. We will keep the names for the morphisms. As pullbacks over nonempty diagrams work in $\mathbf{FinSet}^*_*$ like in **Set** we can construct a pullback. Now we must construct the dashed arrows that satisfy the dual properties of Lemma 3.1.13.

$$\begin{tikzcd}
D \arrow[r, tail, "b^*c"] \arrow[d, "c^*b"] \arrow[dr, phantom, very near start, "\lrcorner"] & B \arrow[d, "b"'] \\
C \arrow[u, dashed, bend left, tail, "(c^*b)'"] \arrow[r, tail, "c"] & A \arrow[u, dashed, bend right, tail, "b'"'] \arrow[l, dashed, bend left, tail, "c'"]
\end{tikzcd}$$

Let c', b' and $(cb)'$ be retractions of c, b and c^*b that sends everything that is not in their image to 0. We will show the required commutativity by case distinction Let $x \in C$. Note that $x \in \operatorname{im} c^*b$ if and only if $c(x) \in \operatorname{im} b$ as this is a pullback diagram. Thus it is immediately clear that if x is not contained in $\operatorname{im} c^*b$ that $(b^*c)(c^*b)'(x) = 0 = b'c(x)$. Let us turn to the case that $x \in \operatorname{im} c^*b$. As restricted to those x the map c^*b is surjective it suffices to check $b'c(c^*b) = (b^*c)(c^*b)'(c^*b)$.

$$(b^*c) = b'b(b^*c) = b'c(c^*b) = (b^*c)(c^*b)'(c^*b) = (b^*c)$$

Therefore the corresponding diagram in $\square$ fulfills the requirements of Lemma 3.1.13, and is thus an absolute pushout square. $\square$

Next we start looking at the presheaf topos on our cube category $\widehat{\square}$. We will recall some important concepts and give some constructions, that we will use later. We also introduce some notation.

3.1.17 Definition (Notation for Cubical Sets). We fix the following notations for Cartesian cubical sets:

- $* := \text{よ}[0]$
- $\mathbb{I} := \text{よ}[1]$
- $\delta_k := \text{よ}\mathrm{d}_k$ with $k \in \{0, 1\}$
- $\delta := \langle \mathsf{id}_{\mathbb{I}}, \mathsf{id}_{\mathbb{I}} \rangle : \mathbb{I} \to \mathbb{I} \times \mathbb{I}$

Figure 2: A square degenerated with a connection. The upper and right side are equal while the left and bottom side is degenerated to point.

3.1.2 Dedekind cubes

We will also encounter the so-called Dedekind cubes. As they are not in the focus of this discussion, we give the needed definitions, but omit most of the proofs of their properties. There are also multiple ways to look at them. We will present multiple definitions, but skip the equivalence proofs as they don't bear much insight into the subject.

3.1.18 Definition.

- Let **FL** be the category of finite lattices and monotone maps.
- Let $\mathbb{2}$ be the finite lattice $\bot \leq \top$.

3.1.19 Definition. The category $\square_{\wedge\vee}$ is the full subcategory of **FL**, restricted to objects of the form $\mathbb{2}^n$ with $n \in \mathbb{N}$.

3.1.20 Definition. The category $\square_{\wedge\vee}$ is the Lawvere-theory of the theory of lattices.

Unraveling this definition gives us a quite similar description to our description of Cartesian cubes. Let F be the functor that assigns to a set its free lattice. We then have as objects natural numbers (or up to equivalence finite sets). And again we will write elements of these as lower roman letters instead of natural numbers to keep the confusion between sets and their elements minimal. A morphism $m \to n$ is a function from $n \to F(m)$. This adds a few more degeneracy maps, but the general geometric intuition stays the same. The elements of an object are dimensions, and maps do give us for every dimension in the target a term, that describes the behavior of our domain in that particular direction. If we look at an example again one might ask what the difference is, between the new degeneracies map from a square to the interval in comparison to the old one. We can view these as degenerate squares where we map the top and right (or bottom and left) side along the interval and the other to sides to $\bot$ (or $\top$). This wasn't possible before and will make a drastic difference once we start considering lifting problems.

3.1.21 Proposition. The Dedekind cubes are an EZ-category

Proof sketch: This can be proven by an analogous argument as theorem 3.1.12 $\square$

If it is clear from the context we will use the notation from definition 3.1.7 and definition 3.1.17 for the analogous objects of the Dedekind cubes.

4 Model structure

4.1 Equivariant Cartesian Cubes

We now present our, main object of study, the equivariant model structure on cubical sets. We will give a description of the two factorization systems involved and then follow an argument from [CS22], to show that it is indeed a model structure. The model structure as well as the presentation is taken from [Rie20]. This equivariant model structure is also know as the ACCRS model structure.

The "usual" model structure on this cube category is not equivalent to spaces, as shown by Buchholtz. If we build the quotient by swapping the dimensions of a square, the resulting space is not contractible, while in spaces it is. The idea is to correct this defect, by making the embedding of the point a trivial cofibration. This is archived by forcing an extra property onto the fibrations.

$$\begin{tikzcd}
* \arrow[r, "\mathrm{id}"] \arrow[d, tail, "f"] & * \arrow[r] \arrow[d, tail, "\sigma f"'] & * \arrow[r, "\beta"] \arrow[d, tail] & X \arrow[d, "f"] \\
\mathbb{I}^2 \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "\sigma"'] \arrow[urrr, dashed, "j(\alpha e\sigma,\beta)"] & \mathbb{I}^2 \arrow[r, "e"] \arrow[urr, dashed, "j(\alpha e,\beta)"'] & Q \arrow[r, "\alpha"] & Y
\end{tikzcd}$$

Here σ is the map that swaps the dimensions and e is the coequalizer map. The hope would be, that the coequalizer would now present a lift for the right most lifting problem. But in general that does not hold, the lifts neither have to commute with σ nor with id. The path forward will be to restrict the fibrations to have the desired property.

4.1.1 Cofibrations and trivial fibrations

The short way of specifying this weak factorization system is by saying the cofibrations are the monomorphisms. Another, longer way, but for the further development more enlightening, is formulating this as a uniform lifting property. It is also not equivalent from the viewpoint of an awfs, as this definition has extra conditions on the chosen lifts. The condition to have uniform lifts comes from [CCHM] and is generalized in [GS17] . This requirement is motivated to have constructive interpretation of Cubical type theory. As we don't focus on the type theoretic interpretation but on the model structure, we won't explore this further.

4.1.1 Definition. Let $J : \mathcal{M} \rightarrowtail \mathcal{A}^{\to}$ be subcategory which objects are a pullback stable class of morphisms in $\mathcal{A}$, and morphisms are the Cartesian squares between them. The category $J^{\boxslash}$ is called the category *uniform right lifting morphisms* with respect to $\mathcal{M}$. Objects of $J^{\boxslash}$ are said to have the *uniform right lifting property*.

4.1.2 Definition. The category of *uniform generating cofibrations* has as objects monomorphisms into a cube $C \rightarrowtail \mathbb{I}^n$ of arbitrary dimension, and as morphism Cartesian squares between those.

4.1.3 Definition. A *uniform trivial fibration* is a right map with respect to the inclusion functor from the category of uniform generation cofibrations into $\widehat{\square}^{\to}$.

From this, it is not immediately clear that the left maps are all monomorphisms, but it follows from [GS17, Proposition 7.5]. By Garners small object argument Theorem 2.1.11, this gives rise to an awfs, which we call (TC, F), and an underlying wfs, that we call $(\mathcal{C}_t, \mathcal{F})$.

4.1.2 Trivial Cofibrations and Fibrations

As sketched above, the strategy will be to make more maps trivial cofibrations. This is done by making it harder to be a fibration. Before we give the definition in full generality, we need to address which coequalizer we talk about precisely.

Of course, doing this only in the 2 dimensional case is not enough. For this, we need to say what the "swap" maps are. What these should do, is permuting the dimensions. So let Σ_k be the symmetric group on k elements. For every $\sigma' \in \Sigma_k$, we get a map from $\sigma : \mathbb{I}^k \to \mathbb{I}^k$, by $(\pi_{\sigma'(1)}, \ldots, \pi_{\sigma'(k)})$, where $\pi_l : \mathbb{I}^k \to \mathbb{I}$ is the l-th projection. Also if we have a sub-cube $f : \mathbb{I}^n \rightarrowtail \mathbb{I}^k$, we get a pullback square of the following form

$$\begin{tikzcd} \mathbb{I}^n \arrow[r, "\mathrm{id}"] \arrow[d, tail, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathbb{I}^n \arrow[d, tail, "\sigma f"] \\ \mathbb{I}^k \arrow[r, "\sigma"] & \mathbb{I}^k \end{tikzcd}$$

whose right arrow we call σf.

Now we describe our generating trivial cofibrations. We recall example Section 4.1. We would like to do the same here, but a bit more general. There is also another problem we didn't talked about yet. In **Top**, if we start with a compact space X, and let $\partial X \to X$ be its boundary inclusion. We can then not only lift against the cylinder filling problem with the cap at the bottom $\delta_0 \mathbin{\widehat{\otimes}} (\partial X \to X)$, or the at the top $\delta_0 \mathbin{\widehat{\otimes}} (\partial X \to X)$, but at every point in the interval. Even more we can prove the existence of a lift leaving the point as a variable. In **Top**, $\widehat{\Delta}$, and $\widehat{\square_{\wedge\vee}}$ this follows automatically from the endpoint liftings. In the absence of connections this is not true. So we need to accomedate for that and make this part of our generating trivial cofibrations.

To solve this problem we are going for box-fillings in the context of a cube. The definition of those will be very similar we will just work in the slice over a cube $\mathbb{I}^k$. We do the pushout-product in the slice, and forget the slice, to get a map in $\widehat{\square}$.

$$\left(\widehat{\square}/_{\mathbb{I}^k}\right)^{\rightarrow} \times \left(\widehat{\square}/_{\mathbb{I}^k}\right)^{\rightarrow} \xrightarrow{\hat{\times}_{\mathbb{I}^k}} \left(\widehat{\square}/_{\mathbb{I}^k}\right)^{\rightarrow} \xrightarrow{\text{forget}} \widehat{\square}^{\rightarrow}$$

What do we change by working in the slice? For intuition let us first look at the case with a one dimensional context. The point in this context is the terminal object in $\widehat{\square}/_{\mathbb{I}}$. The interval in this cube category could be described as the pullback, of the interval in $\widehat{\square} = \widehat{\square}/_{\mathbb{I}^0}$, along the unique map $\mathbb{I} \to \mathbb{I}^0$, or in simpler words it is the left projection $\pi_l : \mathbb{I} \times \mathbb{I} \to \mathbb{I}$. Let us now look what maps we have from the point into the interval $\mathsf{id}_{\mathbb{I}} \to \pi_l$. We get the two endpoint maps we expected, as $\langle \mathsf{id}, 0\rangle$ and $\langle \mathsf{id}, 1\rangle$, but in this context we get an additional map, namely $\langle \mathsf{id}, \mathsf{id}\rangle$. If we would forget the context again (aka, being in the slice), this is the diagonal of the square. From a type theoretic perspective, this is not really surprising. Here it amounts to the question, how many different terms of type $\mathbb{I}$ can be produced. In an empty context the answer is two.

$$\vdash 0 : \mathbb{I} \qquad\qquad \vdash 1 : \mathbb{I}$$

But in the context of $\mathbb{I}$, the answer is three.

$$i : \mathbb{I} \vdash 0 : \mathbb{I} \qquad\qquad i : \mathbb{I} \vdash 1 : \mathbb{I} \qquad\qquad i : \mathbb{I} \vdash i : \mathbb{I}$$

Like the type theory suggest, we can think of this, as generic element of the interval. That this element that ranges across the interval. As we sketched above we want to be able to fill at the point $i : \mathbb{I}$ as well, this amounts to being able to have a cube filling property along those diagonals, or in other words we want them to be trivial cofibrations too. The earliest mention of the idea that I am aware of, is this short note [Coq14], for a discussion see for example [Ang+21; Awo23a].

The generalization of the boundary inclusion is straight forward. We just take any cube over $\mathbb{I}^k$ $\zeta : \mathbb{I}^n \to \mathbb{I}^k$ and a monomorphism $c : C \rightarrowtail \mathbb{I}^n$, this also induces an object in the slice by composition. Notice that we did not require that $n \geq k$. If we pack all of this into a definition we get the following.

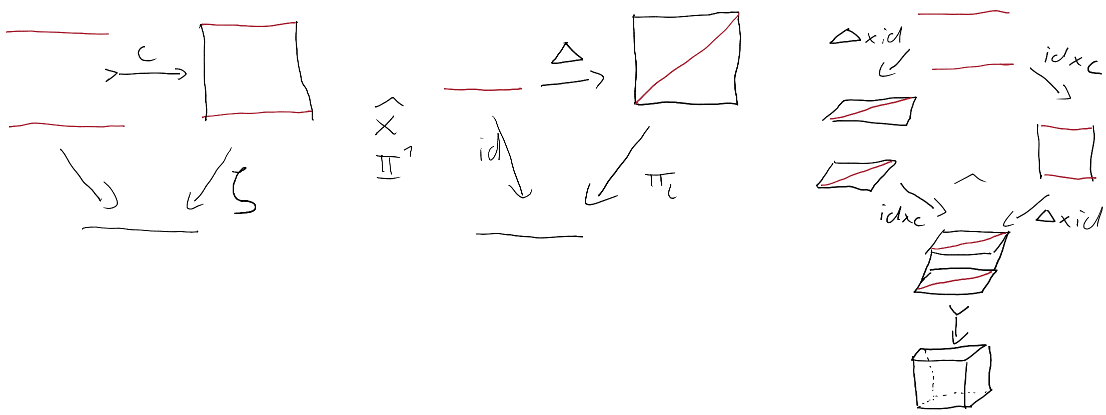

Figure 3: An example construction of a trivial cofibration

4.1.4 Definition. Let $\zeta : \mathbb{I}^n \to \mathbb{I}^k$ be a cube over $\mathbb{I}^k$ and $c : C \rightarrowtail \mathbb{I}^n$ monic. A map of the form

$$\begin{tikzcd} C \arrow[r, tail, "c"] \arrow[dr] & \mathbb{I}^n \arrow[d, "\zeta"] \arrow[r, phantom, "\hat{\times}_{\mathbb{I}^k}"] & \mathbb{I}^k \arrow[r, "\langle \mathrm{id}, l \rangle"] \arrow[d, "\mathrm{id}"] & \mathbb{I}^k \times \mathbb{I}^m \arrow[dl, "\pi_l"] \\ & \mathbb{I}^k & \mathbb{I}^k & \end{tikzcd}$$

is called *generating trivial cofibration*, as a map in $\widehat{\square}$ with signature $B_{c,\zeta,l} \rightarrowtail \mathbb{I}^m \times \mathbb{I}^n$.

We now want to define our fibrations as the uniform equivariant right lifting maps. To do this, we need to define what this actually means. Uniformity wants a choice of lifts that agree with pullback squares between (trivial) cofibrations.

$$\begin{tikzcd} A \arrow[r, "a"] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[r, "x"] \arrow[d, tail] & X \arrow[d, two heads] \\ \mathbb{I}^k \arrow[r, "b"'] \arrow[urr, dashed, "j(xa,yb)"] & \mathbb{I}^n \arrow[r, "y"'] \arrow[ur, dashed, "j(x,y)"'] & Y \end{tikzcd}$$

The uniformity condition wants that the triangle containing both chosen lifts commutes, or as an equation $j(x, y)b = j(ax, yb)$. We now need to combine the uniformity and the equivariance condition. As the pushout-product is functorial, it is enough to describe these conditions between the building blocks of our trivial cofibration and apply the pushout-product again. Again we will do this in a slice and forget the slice afterwards.

$$
\left(\begin{tikzcd}
D \arrow[r, "a"] \arrow[d, tail, "\partial"] \arrow[dr, phantom, "\lrcorner", very near start] & C \arrow[d, tail, "c"] & \\
\mathbb{I}^m \arrow[r, "\alpha"] & \mathbb{I}^n \arrow[dr, "\zeta"] & \\
& & \mathbb{I}^k
\end{tikzcd}\right)
\hat{\times}_{\mathbb{I}^k}
\left(\begin{tikzcd}
\mathbb{I}^k \arrow[r, "\mathsf{id}"] \arrow[d, "\langle \mathsf{id}, l\rangle"] & \mathbb{I}^k \arrow[d, "\langle \mathsf{id}, \sigma l\rangle"] & \\
\mathbb{I}^k \times \mathbb{I}^l \arrow[r, "\mathsf{id} \times \sigma"] & \mathbb{I}^k \times \mathbb{I}^l \arrow[dr, "\pi_l"] & \\
& & \mathbb{I}^k
\end{tikzcd}\right)
$$

Notice that we understand the vertical morphisms as objects of $\widehat{\square}^{\to}$ and the horizontal ones as morphism in $\widehat{\square}^{\to}$. Also notice that the left diagram is a square which captures our uniformity condition and the right one captures equivariance. This is interesting, because this means that uniformity is a condition that can be stated on the base of the filling, while equivariance is a condition on the cylinder walls, that are now multi dimensional. In other words we added a group action on the products of intervals. Also notice that the pushout-product of the columns yield trivial cofibrations. We will denote the resulting commutative square as

$$
\begin{tikzcd}
B_{\partial,\zeta\alpha,l} \arrow[r, "\langle \alpha, a\times\sigma\rangle"] \arrow[d, tail] & B_{c,\zeta,\sigma l} \arrow[d, tail] \\
\mathbb{I}^m \times \mathbb{I}^l \arrow[r, "\alpha\times\sigma"] & \mathbb{I}^n \times \mathbb{I}^l
\end{tikzcd}
\tag{2}
$$

Now we can finally give the definition of uniform equivariant fibrations.

4.1.5 Definition. The category of *generating uniform equivariant trivial cofibrations* has as objects generating cofibrations and as morphism squares of the shape defined in Eq. (2) above.

4.1.6 Definition. A *uniform equivariant fibration* is a right map with respect to the inclusion functor of the category of generating uniform equivariant trivial cofibrations into $\widehat{\square}^{\to}$.

4.1.7 Remark. This gives us directly that all quotients of representables are contractible, because it is now part of the definition that the point inclusion is a weak equivalence.

4.1.8 Remark. There are more ways to state the equivariance property. This one follows closely [Rie20]. We could also passed to a category of group actions on $\widehat{\square}$ and defined uniform fibrations in a naive way in that category. This construction will show up later Remark 4.1.31, also see [ACCRS, Section 4 & 5] for more details.

4.1.9 Notation. By Theorem 2.1.11 these categories of maps form an awfs. We denote it by (TC, F) and the underlying wfs $(\mathcal{C}_t, \mathcal{F})$

4.1.3 The Premodel Structure of Equivariant Cubical Sets

On our way to show that this is a model structure, we will use the theory of premodel structures [Bar19; CS22, section 3; ACCRS, section 3; Sat].

4.1.10 Definition ([Bar19; CS22, def. 3.1.1])**.** A *premodel Structure* on a finitely co-complete and complete category $\mathcal{M}$ consists of two weak factorization systems (C, F_t) (the *cofibrations* and *trivial fibrations*) and (C_t, F) (the *trivial cofibrations* and *fibrations*) on $\mathcal{M}$, such that $C_t \subseteq C$ (or equivalently $F_t \subseteq F$).

4.1.11 Remark. This structure ascends to all slices and is created by the corresponding forgetful functor. This should not be surprising, as model structures do the same.

As all trivial cofibrations are monomorphisms, we immediately get that the two defined factorization systems above form a premodel structure. Every premodel structure comes equipped with a notion of weak equivalences.

4.1.12 Definition ([CS22, Definition 3.1.3])**.** We say that a morphism in a premodel structure is a weak equivalence if it factors as a trivial cofibration followed by a trivial fibration; we write $W(C, F)$ for the class of such morphisms.

What is missing, is that these equivalences actually satisfy the 2-out-of-3 condition. The machinery of premodel structures gives us a nice condition to check if we are actually dealing with a model structure. But first, we investigate a bit more structure of our premodel structure.

4.1.13 Definition ([CS22, Definition 3.2.1])**.** A *functorial cylinder* on a category E is a functor $\mathbb{I} \otimes (-) : E \to E$ equipped with endpoint and contraction transformations. Fitting in a diagram as shown:

$$\begin{tikzcd}
\mathsf{Id} \arrow[r, dashed, "\delta_0 \otimes (-)"] \arrow[dr, "\mathsf{id}"'] & \mathbb{I} \otimes (-) \arrow[d, dashed, "\varepsilon \otimes (-)"] & \mathsf{Id} \arrow[l, dashed, "\delta \otimes (-)"'] \arrow[dl, "\mathsf{id}"] \\
 & \mathsf{Id} &
\end{tikzcd}$$

An *adjoint functorial cylinder* is a functorial cylinder such that $\mathbb{I} \otimes (-)$ is a left adjoint.

We can see immediately that the functor $\mathbb{I} \times (-)$, the product with the interval, is a functorial cylinder, and by $\widehat{\square}$ being a presheaf topos even an adjoint functorial cylinder. But for now it is not at all clear that this functor is relevant in a homotopical sense. This is captured by the next definition.

4.1.14 Definition ([CS22, Definition 3.2.5]). We write $\partial = [\delta_0, \delta_1] : \mathsf{Id} + \mathsf{Id} \Rightarrow \mathbb{I} \otimes (-)$ for the inclusion of both endpoints. A *cylindrical* premodel structure on a category E consists of a premodel structure and an adjoint functorial cylinder on E that are compatible in the following sense:

- $\partial \mathbin{\widehat{\otimes}} (-)$ preserves cofibrations and trivial cofibrations,
- $\delta_k \mathbin{\widehat{\otimes}} (-)$ sends cofibrations to trivial cofibrations for $k \in \{0, 1\}$.

These properties are verified quickly, the map $\partial : 1 + 1 \to \mathbb{I}$ is monic and thus it follows from Proposition 2.2.6. We check the second property on generating cofibrations, Plugging in the definition of our functorial cylinder, we see that this is by definition a trivial cofibration if we set $k = 0$ and $m = 1$ in Definition 4.1.4. There is also an even stronger notion.

4.1.15 Definition ([compare HR24]). We say a premodel structure is *generated by an interval*, if we have an interval object $\mathbb{I}$, the functorial cylinder is given by $\mathbb{I} \mathbin{\widehat{\otimes}} (-)$, we have a generic point of the interval $\delta = \langle \mathsf{id}, \mathsf{id} \rangle : \mathbb{I} \to \mathbb{I} \otimes \mathbb{I}$ as a map in the slice over $\mathbb{I}$, and a map is a fibration if and only if $\delta \Rightarrow f$ a trivial fibration.

Sadly our premodel structure is not generated by $\mathbb{I}$. But the appropriate premodel structure of group actions on $\widehat{\square}$ is, see Remark 4.1.31, and for more details [ACCRS, Section 4].

The theory of premodel structures also gives a criterea to determine if a premodel structure actually induces a model structure.

4.1.16 Definition ([CS22, Definition 3.3.3]). We say a premodel category $\mathcal{M}$ has the fibration extension propertyi, when for any fibration $f : Y \to X$ and trivial cofibration $m : X \to X'$, there exists a fibration $f' : Y' \to X'$, whose base change along m is f:

$$\begin{tikzcd} Y \arrow[r, dashed] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & Y' \arrow[d, dashed, "f'"] \\ X \arrow[r, "m"] & X' \end{tikzcd}$$

4.1.17 Theorem ([CS22, Theorem 3.3.5]). Let $\mathcal{M}$ be a cylindrical premodel category in which

- all objects are cofibrant;
- the fibration extension property is satisfied.

Then the premodel structure on $\mathcal{M}$ defines a model structure.

The first item is fast verified, as all monomorphisms are cofibrations. The second condition we won't show directly. We will get it for free if we construct universes, which we want to do anyway. Before we construct universes we will first talk about another property that follows from general pre model structure theory. This property relates back to type theory, as it is the categorical side of glue types. These keep track of type extension and are used to prove univalence. We will use it later to show that the base of our constructed universe is fibrant. In categorical terms it says the following.

4.1.18 Definition ([Sat17])**.** A cylindrical premodel structure has the equivalence extension property when any weak equivalence e over an object A can be extended along any cofibration $i : A \rightarrowtail B$ to a weak equivalence f over B with a specified codomain extending that of the original map:

$$\begin{tikzcd}
X_0 \arrow[rr, dotted] \arrow[dr, "e"] \arrow[ddr, "p_0"'] & & Y_0 \arrow[dr, dashed, "f"] \arrow[ddr, dotted] & \\
& X_1 \arrow[rr] \arrow[d, "p_1"] \arrow[drr, phantom, "\lrcorner", very near start] & & Y_1 \arrow[d, "q_1"] \\
& A \arrow[rr, tail, "i"] & & B
\end{tikzcd}$$

In a setting such as a presheaf topos where we have universe levels, there is an additional requirement: for sufficiently large inaccessible cardinals $\varkappa$, if p_0, p_1, and q_1 are $\varkappa$-small, so is the extended fibration.

We could have equivalently used [Sat17, Proposition 5.1], or [Awo23a, Section 7]. This version aligns more direct with the data that we are given.

4.1.19 Theorem ([ACCRS, Theorem 3.3.3])**.** Let E be a locally Cartesian closed category with a cylindrical premodel structure in which the cofibrations are the monomorphisms, and these are stable under pushout-products in all slices. Then the equivalence extension property holds in E.

The pushout product claim is verified by Proposition 2.2.6, and as every topos is cartesian closed, we can apply the theorem.

4.1.4 Universes for Fibrations

As these tools can get quite technical and tedious to verify, we will only present the idea how these constructions work and refer for the technical details to the source material.

The general construction of these universes follows mostly the same idea. We lift a Grothendieck universe from **Set** to the desired presheaf category, obtaining a Hofmann-Streicher universe [HS97], classifying small maps. This works in general for any presheaf topos over a small site. Afterwards, we restrict this universe to only small fibrations. We are then faced with a challenge, we need to prove that the resulting classifier is itself a fibration. In some cases this can be made easier or avoided by stating this construction in a specific way. For example by the arguments in [Awo23b, Section 7] it suffices to construct *classifying types*.

4.1.20 Definition. A *classifying type* for an awfs is a construction Fib such that for every object X and every morphism $f : A \to X$, there is a map $\mathrm{Fib}(f) : \mathrm{Fib}(A) \to X$ and the sections are in correspondence with R-algebra structures on f. Given a map $g : Y \to X$, then Fib is stable under pullbacks along g.

$$\begin{tikzcd}
g^*A \arrow[r] \arrow[d, "g^*f"'] \arrow[dr, phantom, "\lrcorner", very near start] & A \arrow[d, "f"] \\
Y \arrow[r, "g"] & X \\
\mathrm{Fib}(g^*A) \arrow[r] \arrow[u, "\mathrm{Fib}(g^*f)"] \arrow[ur, phantom, "\urcorner", very near start] & \mathrm{Fib}(A) \arrow[u, "\mathrm{Fib}(f)"']
\end{tikzcd}$$

Remark. We might think of this as an internalization of the R-algebra structures on f.

In preparation of the fibrant case we will first classify the trivial fibrations. We don't have any equivariance condition on them,thus they are the same as in [Awo23a, Section 2], when if we assume $\Phi = \Omega$.

Let us first think about the easiest case where the domain of the trivial fibration is the terminal. We should ask, what the TF-algebra structures are. These are choice functions of uniform liftings against cofibrations. So let us consider them first. If we picture a lifting problem,

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail] & X \arrow[d] \\
C \arrow[r] & *
\end{tikzcd} \tag{3}$$

we might think of this as a filling porblem of some shape A in X to some shape C in X. Or in other words as a complition of a partial map[2] from $C \to X$. We

[2]Recall that a partial map $C \to X$ is a span $C \leftarrow A \to X$, where the left map is a monomorphism.

can classify these by the partial map classifier, given by $X^+ \coloneqq \Sigma_{\varphi:\Omega} X^{[\varphi]}$ [GK13]. This gives us an object X^+ (technicaly equipped with a map into the terminal) and a monomrphism $\eta_X : X \rightarrowtail X^+$ such that for every partial map $C \to X$ there is a unique map $a : C \to X^+$, completing the following pullback square.

$$\begin{tikzcd}
A \arrow[r] \arrow[d, tail] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "\eta_X"] \\
C \arrow[r, "a"] & X^+
\end{tikzcd}$$

This makes $(-)^+$ into a pointed endofunctor, and with little surprise the +-algebras are in correspondence with the TF-algebras.

4.1.21 Proposition ([Awo23a, Proposition 15]). The algebras of the pointed endofunctor $(-)^+$ are in correspondence to the algebras of the pointed endofunctor TF with codomain $*$.

Proof sketch: A +-algebra is an object X together with a morphism $\alpha : X^+ \to X$ such that $\alpha\eta_x = \mathrm{id}$. Take some lifting problem like Eq. (3), define the chosen map to be αa. If we are given a chosen lift, we compose it with η_X to retrieve a. $\square$

The benefit of $(-)^+$-algebras is, that they are morphisms in $\widehat{\square}$ rather than in $\widehat{\square}^{\to}$. We can internalize the condition to be a $(-)^+$-algebra by means of the internal Hom.

$$\begin{tikzcd}
+\text{-Alg}(X) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & {[X^+, X]} \arrow[d, "{[\eta_X, \mathrm{id}_X]}"] \\
* \arrow[r, "\lambda \mathrm{id}_X"] & {[X, X]}
\end{tikzcd}$$

It is immediatly clear that a section of the left map correspond to the desired +-algebra structure.

If we want to archive the same with a trivial fibration that has arbitrary codomain Y, we might just move to the slice topos $\widehat{\square}/_Y$, there a map to Y map becomes again a map into the terminal. To not be confused we will denote the partial map classification functor in the slice over Y by the name $(-)^{+Y}$. The algebras will be called relative +-algberas. We have even more luck and this functor commutes with the pullback functor $\widehat{\square} \to \widehat{\square}/_Y$,

$$\begin{tikzcd}
\widehat{\square}/_Y \arrow[r, "(-)^{+Y}"] & \widehat{\square}/_Y \\
\widehat{\square} \arrow[r, "(-)^+"] \arrow[u, "Y^*"] & \widehat{\square} \arrow[u, "Y^*"]
\end{tikzcd}$$

which allows us to compute it fiberwise [Awo23a, Proposition 12]. This allows us to write $(A)^{+Y} = \Sigma_{y:Y} A_y^+$, where A_y is the fiber of A at y. We will still continue to write $+$-Alg, if there is no ambiguity. As all the involved functors $(-)^+$, pullbacks and exponentials are pullback stable so is $+$-Alg. This lets us conlude the following.

4.1.22 Proposition. $+$-Alg is a classifying type for trivial cofibrations.

We will move down a very similar path as [Awo23a] but need to keep track of the group actions of the symmetric groups on representables. For this, we will pass to a category that [ACCRS] calls cubical species. The argument in essence will be very similar to the argument there, though in a different language.

4.1.23 Definition (compare [ACCRS, Section 4])**.** By abuse of notation we will say Σ_k to be the one object groupoid induced by the symmetric group Σ_k. Let $\Sigma := \coprod_{k \in \mathbb{N}} \Sigma_k$. We define $\widehat{\square}^{\Sigma}$ to be the category of *cubical species*. For a cubical species A we write A_k for the cubical set in the image of Σ_k.

4.1.24 Remark. $\widehat{\square}^{\Sigma}$ is a presheaf topos, regarded as $\mathbf{Set}^{\Sigma \times \mathbf{FinSet}_*^*}$.

4.1.25 Remark. [ACCRS] excludes the case for $k = 0$, because they want to exhibit a model category on cubical species. For the construction on cubical Sets this makes no difference, compare [ACCRS, Remark 4.3.17].

We need to define the interval, and the generic point inclusion. Recall that a functor from a group into a category is the same as choosing an object of that category together with a group action on it.

4.1.26 Definition ([ACCRS, Example 4.3.5])**.** Let $\mathrm{I} : \Sigma \to \widehat{\square}$ be the functor that sends Σ_k to $\mathbb{I}^k$ together with the group action that freely permutates the dimensions. Spelled out this means the if $\mathbb{I}^k$ is represented by $A = \{\bot | x_1, \dots, x_k | \top\}$, then an element $\sigma \in \Sigma_k$ is send to the automorphism $A \mapsto \{\bot | x_{\sigma(1)}, \dots, x_{\sigma(k)} | \top\}$.

4.1.27 Remark. This is exactly the same group action we want our liftings be equivariant under, compare Remark 4.1.8.

4.1.28 Remark ([ACCRS, Lemma 4.2.7])*.* The interval I is tiny.

4.1.29 Definition ([ACCRS, Definition 4.3.3])**.** The diagonal $\delta : \mathrm{I} \to \mathrm{I} \times \mathrm{I}$ as a map in the slice $\widehat{\square}^{\Sigma} /_{\mathrm{I}}$ is called the *generic point of* I.

4.1.30 Remark. Our classifcation induce a premodle structure on cubical species, where the cofibrations are all monomorhpisms, the trivial fibrations are classified by relative $+$-algebras and the fibrations are those maps such that their pullback to the slice over I are getting sent to a trivial fibration by $\delta \Rightarrow (-)$. Premodel structures in this way are called *generated by* I.

4.1.31 Remark. By general abstract nonesense the premodel structure from Remark 4.1.30 is an adjoint (because topoi are cartesian closed) cylindrical premodel structure.

4.1.32 Remark. By Theorem 4.1.19 this premodel structure also fulfills the equivalence extension property.

We will need another property that again follows from general premodel structure theory. We will not get this immediately for equivariant cubes, as these are not generated by an interval[3]. Namely that left maps are pullback stable along right maps, for both factorization systems.

4.1.33 Theorem ([Awo23a, §5; HR24; ACCRS, Proposition 3.4.2])**.** Let E be a locally cartesian closed category with a premodel structure in which the cofibrations are the monomorphisms. Suppose it is generated by an interval, then the premodel structure satisfies the Frobenius condition.

The main idea to continue classifying uniform fibrations in $\widehat{\square}^{\Sigma}$, from here on out is quite simple. As being a uniform fibration is equivalent by being send to a trivial fibration by the right adjoint of the Leibniz application functor of the interval, we get that a $+$-algebra structure on $\delta \Rightarrow f$ is equivalent to an F-algebra structure on f. This must then be reformulated to get the desired map. This reformulation is quite tedious and so we will refer for the details to [Awo23a, The classifying types of unbiased fibration structures]. In the source this takes place in $\widehat{\square}$, but it only makes use of the features we already established for $\widehat{\square}^{\Sigma}$. An important detail to this procedure is, that in order to get the resulting classifying type pullback stable, this argument uses that the interval is tiny.

We get a pullback stable construction $\mathrm{Fib}^{\Sigma}(f)$, whose sections now correspond componentwise to F-algebra structures, that are equivariant with respect to the group action of Σ^k.[4]

We are now going to extract the classifying type for equivariant cubical fibrations. For this we need to relate cubical sets and cubical species. There exists the constant diagram functor Δ that sends a cubical set to the constant cubical species.

Since $\widehat{\square}$ is complete this functor has a right adjoint[5] Γ compare [ACCRS, Section 5.1], which is given by

$$\Gamma(A) := \prod_{k \in \mathbb{N}} (A_k)^{\Sigma_k}$$

[3]we could infer it from cubical species

[4]This can be made into a classifying type for some awfs compare [ACCRS, Section 4.].

[5]and also a left adjoint as $\widehat{\square}$ is co-complete.

where $(A_k)_k^{\Sigma}$ is the cubical set of Σ_k fixed points. It is not hard to see, that this is indeed the right adjoint functor.

4.1.34 Proposition. We have a pair of adjoints $\Delta \dashv \Gamma$

Proof. We need to show

$$\mathrm{Hom}_{\widehat{\square}^{\Sigma}}(\Delta(A), B) = \mathrm{Hom}_{\widehat{\square}}\left(A, \prod_{k \in \mathbb{N}} (B_k)^{\Sigma_k}\right)$$

But this is immediatly clear: on the right hand side, we have componentwise equivariant natural translations from $\Delta(A)_k = A$ to B_k. They can only be valued in the fixed points in B_k as $\Delta(A)_k$ carries the trivial group action. A map on the right side is also a collection of natural transformations from A to the fixed points of B_k. $\square$

As the sections of the classifying types in $\widehat{\square}^{\Sigma}$, correspond componentwise to choice functions of lifts in $\widehat{\square}$, for different dimensional box fillings, and they are equivariant by virtue of being a morphism in $\widehat{\square}^{\Sigma}$, we get now that an F-algebra on $f : A \to B$ structure corresponds to a section.

$$\begin{tikzcd}
& \mathrm{Fib}^{\Sigma_k}(\Delta(A)) \arrow[d, "\mathrm{Fib}^{\Sigma_k}(\Delta(f))"] \\
\Delta(B) \arrow[ur, dashed] \arrow[r, equal] & \Delta(B)
\end{tikzcd}$$

By adjointness, we can transpose this lift to a lift in cubical sets, and by pulling back along the adjunction unit to splits of a map over B (compare [ACCRS, Lemma 5.3.3, 2.1.16]).

$$\begin{tikzcd}
\mathrm{Fib}(B) \arrow[r] \arrow[d, "\mathrm{Fib}(f)"] \arrow[dr, phantom, "\lrcorner", very near start] & \Gamma(\mathrm{Fib}^{\Sigma_k}(\Delta(A))) \arrow[d, "\Gamma(\mathrm{Fib}^{\Sigma_k}(\Delta(f)))"] \\
B \arrow[u, dashed, bend left] \arrow[ur, dashed] \arrow[r, "\eta_B"] & \Gamma\Delta(B)
\end{tikzcd}$$

4.1.35 Remark. From this discussion it is also clear that f is a equivariant fibration in $\widehat{\square}$, if and only if $\Delta(f)$ is a fibration in $\widehat{\square}^{\Sigma}$.

We will now present the idea, how to get fibrant universes from such classifying types. This is mainly taken from [Awo23b, Section 7]. One might hope that we can construct a universal fibration, in the sense that every fibration is a pullback of this universal fibration. This can not happen for size reasons. We will do the next best thing and construct a universe for big enough kardinals κ. Because our set theory has Grothendieck universes, this will give us enough universes to classify all fibrations.

To start we will call our Hofmann-Streicher, or κ-small map classifier $p : \dot{\mathcal{V}}_\kappa \to \mathcal{V}_\kappa$.

4.1.36 Theorem ([Awo23b, Proposition 10]). For a large enough κ, there is a universe for κ-small fibrant maps, in the sense that there is a small fibration $\pi : \dot{\mathcal{U}}_\kappa \to \mathcal{U}_\kappa$ such that every small fibration $f : A \to X$ is a pullback of it along a connonical classifying map a

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{U}}_\kappa \arrow[d, "\pi_\kappa"] \\ \mathcal{X} \arrow[r, "a"] & \mathcal{U} \end{tikzcd}$$

Proof. We can then construct our universe that classifies κ-small fibrations by setting $\mathcal{U}_\kappa \coloneqq \mathrm{Fib}(\dot{\mathcal{V}}_\kappa)$ and building the following pullback.

$$\begin{tikzcd} \dot{\mathcal{U}}_\kappa \arrow[r] \arrow[d, "\pi_\kappa"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}}_\kappa \arrow[d, "p"] \\ \mathcal{U} \arrow[r, "\mathrm{Fib}(p)"] & \mathcal{V} \end{tikzcd}$$

To see that this is a fibration, we will exhibit we exhibit a split of the classifying type of π_κ. As the classifying type is pullback stable we get the following diagram

$$\begin{tikzcd} \dot{\mathcal{U}}_\kappa \arrow[r] \arrow[d, "\pi_\kappa"'] \arrow[dr, phantom, "\lrcorner", very near start] & \dot{\mathcal{V}}_\kappa \arrow[d, "p"] \\ \mathcal{U}_\kappa \arrow[r, "\mathrm{Fib}(p)"] & \mathcal{V}_\kappa \\ \mathrm{Fib}(\dot{\mathcal{U}}_\kappa) \arrow[r] \arrow[u] \arrow[ur, phantom, "\urcorner", very near start] & \mathrm{Fib}(\dot{\mathcal{V}}_\kappa) \arrow[u, "\mathrm{Fib}(p)"'] \end{tikzcd}$$

The lower pullback square is a pullback of $\mathrm{Fib}(p)$ along itself. This map has a split, namely the diagonal. To show that this classifies small fibrations, we consider a small fibration $f : A \to X$. Because it is a small map we get it as a pull back along a canonical map $a : X \to \mathcal{V}$.

$$
\begin{tikzcd}
 & & \dot{\mathcal{U}}_\kappa \arrow[dr] \arrow[dd] & \\
 & A \arrow[ur, dashed] \arrow[rr] \arrow[dd] & & \dot{\mathcal{V}}_\kappa \arrow[dd] \\
\mathrm{Fib}(A) \arrow[rr] \arrow[dr] & & \mathcal{U}_\kappa \arrow[dr] & \\
 & X \arrow[ul, dashed, bend left, "s"] \arrow[ur, dashed, "s'"] \arrow[rr, "a"] & & \mathcal{V}
\end{tikzcd}
$$

We get the lower square as a pullback as Fib is pullback stable and $\mathcal{U} = \mathrm{Fib}(\dot{\mathcal{V}})$. As f is a fibration, we get a split s of the fibration structure. And so we get a unique corresponding map s', which induces the required pullback square. $\square$

4.1.37 Definition ([Awo23a, Definition 94]). A map $P \to X$ is said to be a weak proposition if the projection $P \times_X P \to P$ is a trivial fibration.

$$
\begin{tikzcd}
P^2 \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & P \arrow[d] \\
P \arrow[r] & X
\end{tikzcd}
$$

Remark. Note that if either projection is a trivial fibration, then both are.

4.1.38 Lemma ([Awo23a, Lemma 95]). For any $A \to X$, the classifying type $\mathrm{TFib}(A) \to X$ is a weak proposition. Moreover, the same is true for $\mathrm{Fib}(A) \to X$.

We again will discuss the proof idea, sort out the differences for the equivariant case and refer to the source material for the technical details.

Proof sketch: We first prove the case for TFib and afterwards reduce the Fib case to it. We first note that if a map $A \to X$ is a trivial fibration, so is $\mathrm{TFib}(A) \to X$. This is because if we expand the definition and look at the $+$-algebra structure, the map $\eta_X : X \to X^+$ is always monic.

Now let $f : A \to X$ and consider the following diagram.

$$
\begin{tikzcd}
\mathrm{TFib}(A) \times_X A \arrow[rr] \arrow[dd] & & A \arrow[dd, "f"] & \\
 & \mathrm{TFib}(A) \times_X \mathrm{TFib}(A) \arrow[rr] \arrow[dl] & & \mathrm{TFib}(A) \arrow[dl] \\
\mathrm{TFib}(A) \arrow[rr] & & X &
\end{tikzcd}
$$

Since TFib is stable under pullback we have that $\mathrm{TFib}(\mathrm{TFib}(A) \times_X A)$ is isomorphic to $\mathrm{TFib}(A) \times_X \mathrm{TFib}(A)$, and since the latter has a canonical section $\mathrm{TFib}(A) \times_X A \to \mathrm{TFib}(A) \times_X \mathrm{TFib}(A)$. Therefore $\mathrm{TFib}(A) \times_X A$ is a trivial fibration over $\mathrm{TFib}(A)$. By previous consideration, we see that also $\mathrm{TFib}(A) \times_X \mathrm{TFib}(A)$ is a trivial fibration over $\mathrm{TFib}(A)$.

For Fib, we now trace through the whole construction process, and reduce it to the case above. For the technical details, we refer to the source material. For our deviation in the construction, we need a few additional properties. Namely the constant diagram functor preserves monomorhpisms, which is obviously true, and that taking exponentials with the interval in cubical species preserves monomorhpisms. As Σ has only endomorphisms, this reduces again to a componentwise check in $\widehat{\square}$, where we see this easily being true for the interval, and the rest follows from $\square$ being closed under products. $\square$

4.1.39 Remark. The source has an additional condition, that in our setting is true.

4.1.40 Lemma ([Awo23a, Lemma 96]). The univeres $\mathcal{U}_\kappa$ satisfies *realignment* in the following sense. Given a κ-small fibration $g : A \to X$ and a cofibration $c : C \rightarrowtail X$, let $f_c : C \to \mathcal{U}_\kappa$ classify the pullback $c^*A \to C$. Then there is a classifying map $f : X \to \mathcal{U}_\kappa$ for A with $fc = f_c$.

$$\begin{tikzcd}
c^*F \arrow[rr] \arrow[dr] \arrow[dd] & & \dot{\mathcal{U}}_\kappa \arrow[dd] \\
& A \arrow[ur, dashed] \arrow[dd, "g"] & \\
C \arrow[rr, "f_c"] \arrow[dr, tail, "c"'] & & \mathcal{U}_\kappa \\
& X \arrow[ur, dashed, "f"] &
\end{tikzcd}$$

Proof. First, we extend this diagram by the small map classifier and by realignment for HS-Universes [Awo23b, Proposition 6], we get an extension f_0.

$$\begin{tikzcd}
c^*F \arrow[rr] \arrow[dr] \arrow[dd] & & \dot{\mathcal{U}}_\kappa \arrow[r] \arrow[dd] & \dot{\mathcal{V}}_\kappa \arrow[dd] \\
& A \arrow[urr, dashed] \arrow[dd, "g"] & & \\
C \arrow[rr, "f_c"] \arrow[dr, tail, "c"'] & & \mathcal{U}_\kappa \arrow[r] & \mathcal{V}_\kappa \\
& X \arrow[urr, dashed, "f_0"] & &
\end{tikzcd}$$

Since g is a fibration we get a lift of $f_1 : X \to \mathcal{U}_\kappa$ of f_0 classifying the fibration structure. This gives us the following diagram in the base.

$$\begin{tikzcd}
C \arrow[r, "f_c"'] \arrow[d, tail, "c"] \arrow[rr, bend left, "\mathrm{Fib}(p) f_c"] & \mathcal{U}_\kappa \arrow[r] & \mathcal{V}_\kappa \arrow[d, equal] \\
X \arrow[r, "f_1"] \arrow[rr, bend right, "Fib(p) f_1"'] & \mathcal{U}_\kappa \arrow[r] & \mathcal{V}_\kappa
\end{tikzcd}$$

We can now pull back $\mathrm{Fib}(p)$ along itself and rearrenge the data.

$$\begin{tikzcd}
C \arrow[r, "\langle f_1 c, f_c \rangle"'] \arrow[d, tail, "c"] \arrow[rr, bend left, "fc"] & \mathcal{U}_\kappa \times_{\mathcal{V}_\kappa} \mathcal{U}_\kappa \arrow[r, "\pi_2"'] \arrow[d, "\pi_1"] & \mathcal{U}_\kappa \arrow[d] \\
X \arrow[r, "f_1"] \arrow[rr, bend right, "f_0"'] & \mathcal{U}_\kappa \arrow[r] & \mathcal{V}_\kappa
\end{tikzcd}$$

By Lemma 4.1.38, $\mathrm{Fib}(\dot{\mathcal{V}}_\kappa) = \mathcal{U}_\kappa \to \mathcal{V}_\kappa$ is a weak proposition, this means π_1 is a trivial fibration and we get a lift $f_2 : X \to \mathcal{U}_\kappa \times_{\mathcal{V}_\kappa} \mathcal{U}_\kappa$, by normal lifting properties. Taking $f \coloneqq \pi_2 f_2$ gives another classifying map for the fibration structure, for which $fc = f_c$ as required. $\square$

4.1.41 Lemma ([Awo23a, Proposition 26]). A map $f : Y \to X$ is a fibration in $\widehat{\square}^{\Sigma}$ if and only if the canonical map u to the pullback, in the following diagram, is a trivial fibration.

$$\begin{tikzcd}
Y^{\mathrm{I}} \times \mathrm{I} \arrow[dr, dashed, "u"] \arrow[drr, bend left, "\mathsf{ev}"] \arrow[ddr, bend right, "f^{\mathrm{I}} \times \mathsf{id}_{\mathrm{I}}"'] & & \\
& Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & Y \arrow[d, "f"] \\
& X^{\mathrm{I}} \times \mathrm{I} \arrow[r, "\mathsf{ev}"'] & X
\end{tikzcd}$$

Proof. Let us write out $\delta \Rightarrow f$.

$$\delta \Rightarrow f = \langle f^{\mathrm{I}} \times \mathsf{id}_{\mathrm{I}}, \langle \mathsf{ev}, p_2 \rangle \rangle : Y^{\mathrm{I}} \times \mathrm{I} \to (X^{\mathrm{I}} \times \mathrm{I}) \times_{X \times \mathrm{I}} (Y \times \mathrm{I})$$

We interpolate the above pullback square with another one.

$$\begin{tikzcd}
Y' \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \times \mathrm{I} \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & Y \arrow[d, "f"] \\
X^{\mathrm{I}} \times \mathrm{I} \arrow[r, "\langle \mathsf{ev}, \mathrm{id}_{\mathrm{I}} \rangle"'] & X \times \mathrm{I} \arrow[r, "p_1"'] & X
\end{tikzcd}$$

By the double pullback lemma the outer square is also a pullback and thus $Y' \simeq (X^{\mathrm{I}} \times \mathrm{I}) \times_{X \times \mathrm{I}} (Y \times \mathrm{I})$ and $\delta \Rightarrow f \simeq u$. □

4.1.42 Corollary ([Awo23a, Corollary 27]). *X in $\widehat{\square}^{\Sigma}$ is fibrant if and only if the canonical map u to the pullback, in the following diagram is a trivial fibration.*

$$\begin{tikzcd}
X^{\mathrm{I}} \times \mathrm{I} \arrow[dr, dashed, "u"] \arrow[drr, bend left, "\mathsf{ev}"] \arrow[ddr, bend right, "p_2"'] & & \\
& \mathrm{I} \times X \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & X \arrow[d] \\
& \mathrm{I} \arrow[r] & *
\end{tikzcd}$$

Proof. This is a special case of the lemma above. □

4.1.43 Theorem ([Awo23a, Proposition 120; compare ACCRS, Proposition 5.3.9]). *The base of the universe $\mathcal{U}_\kappa$ is a fibrant object.*

Proof. By Corollary 4.1.42 it suffices to check that the map $u = \langle p_2, \mathsf{ev} \rangle$ in the following diagram in $\widehat{\square}^{\Sigma}$, is a trivial fibration.

$$\begin{tikzcd}
\Delta(\mathcal{U}_\kappa)^{\mathrm{I}} \times \mathrm{I} \arrow[dr, dashed, "u"] \arrow[drr, bend left, "\mathsf{ev}"] \arrow[ddr, bend right, "p_2"'] & & \\
& \mathrm{I} \times \Delta(\mathcal{U}_\kappa) \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \Delta(\mathcal{U}_\kappa) \arrow[d] \\
& \mathrm{I} \arrow[r] & *
\end{tikzcd}$$

To show this we consider the following lifting problem:

$$\begin{tikzcd}
C \arrow[r, "\langle a', ic \rangle"] \arrow[d, tail, "c"] & \Delta(\mathcal{U}_\kappa)^{\mathrm{I}} \times \mathrm{I} \arrow[d, "\langle p_2, \mathsf{ev} \rangle"] \\
Z \arrow[r, "\langle i, b \rangle"'] \arrow[ur, dotted] & \mathrm{I} \times \Delta(\mathcal{U}_\kappa)
\end{tikzcd}$$

Transposing this problem along the exponential product adjunction we get:

$$\begin{tikzcd}
C \arrow[r, "\langle \mathrm{id}_C, ic \rangle"] \arrow[d, tail, "c"] & C \times \mathrm{I} \arrow[d, "c \times \mathrm{id}_I"] \arrow[ddr, bend left, "a"] & \\
Z \arrow[r, "\langle \mathrm{id}_i, i \rangle"] \arrow[drr, bend right, "b"'] & Z \times \mathrm{I} \arrow[dr, dotted] & \\
 & & \Delta(\mathcal{U}_\kappa)
\end{tikzcd} \tag{4}$$

where a is the uncurried versien of a'. We would get such a desired map, if this was a pushout diagram via the coparing $[a, b]$. So let us compare $Z \times \mathrm{I}$ with the pushout.

$$\begin{tikzcd}
C \arrow[r, "\langle \mathrm{id}_C, ic \rangle"] \arrow[d, tail, "c"] & C \times \mathrm{I} \arrow[d, "c \times \mathrm{id}_\mathrm{I}"] \arrow[ddr, bend left, "c \times \mathrm{id}_\mathrm{I}"] & \\
Z \arrow[r] \arrow[drr, bend right, "\langle id_Z, i \rangle"'] & Z +_C (C \times \mathrm{I}) \arrow[dr, dotted, "d"] & \\
 & & Z \times \mathrm{I}
\end{tikzcd}$$

This is exactly the definition of pushout product so we have $d = c \,\hat{\times}\, \delta$, which is a trivial cofibration. And as $\widehat{\square}^\Sigma$ has the equivalence extension property, see Remark 4.1.32, we can extend $[a, b]$ to the desired map in Eq. (4), and with this we get our desired lift. $\square$

4.1.5 From Premodel Structure to Model Structure

We are now ready to verify the last precondition of Theorem 4.1.17.

4.1.44 Proposition. Fibrant universes with realignment imply the fibrant extension property.

Proof. Let $f : A \to X$ be a fibration and $m : X \rightarrowtail Y$ be a cofibration. Let κ be large enough so that f is κ-small. We can classify f by a map $a : X \to \mathcal{U}_\kappa$ and get the following diagram in the base.

$$\begin{tikzcd}
X \arrow[r, "a"] \arrow[d, tail, "m"'] & \mathcal{U}_\kappa \arrow[d] \\
Y \arrow[r] \arrow[ur, dashed, "a'"] & *
\end{tikzcd}$$

As $\mathcal{U}_\kappa$ is fibrant, this gives us a lift a'. We can then get the required fibration by pulling back π_κ along a'

$$\begin{tikzcd}
A \arrow[rr] \arrow[dr, dashed] \arrow[dd, "f"'] & & \dot{\mathcal{U}}_\kappa \arrow[dd] \\
& Y' \arrow[ur] \arrow[dd, dashed] & \\
X \arrow[rr, "f_c"] \arrow[dr, tail, "c"] & & \mathcal{U}_\kappa \\
& Y \arrow[ur, "a'"'] &
\end{tikzcd}$$

making the required diagram a pullback, by the double pullback lemma. □

4.2 Dedekind Cubes

We will also need a description of the model structure on the Dedekind cubes. As it is not our main focus of this thesis, we will describe the premodel structure and give some hints how to go from there, but don't carry out the whole universe construction. It turns out that in the presence of connections we get liftings of a general point and equivariance for free [CS22, Remark 4.2.26 & Corollary 4.2.24]. While the argument that this modelstructure indeed is a model structure constructing universes is somewhat easier, as we can extract it from an internal description in the presheaf topos, by using strategies from [LOPS].

Characterizing cofibrations and trivial fibrations will done as above.

4.2.1 Definition. A *generating cofibration* is a monomorphism into a cube $c : C \rightarrowtail \mathbb{I}^n$

4.2.2 Definition. A *uniform trivial fibration* is a map with the uniform right lifting property against the generating cofibrations

It will be usefull for our later development, to give a characterization in the internal language of the presheaf topos $\widehat{\square}_{\wedge\vee}$. For more details on the interpretation of the internal semantics of cubical sets see [AGH21]. For the this style of modeling cubical type thoery inside of topoi see [OP18]. We will first give the characterization, then try to give some intuition why it makes sense.

$$\mathrm{TFib}\, A \coloneqq \Pi_{\varphi:\Omega}\Pi_{f:[\varphi]\to A}\Sigma_{a:A}\Pi_{\alpha:[\varphi]} \upsilon(\alpha) = a \tag{5}$$

So assume we have a type A in some context Γ. For the discussion we will identify types with their respecting display maps, to keep the notational overhead low. In this context the display map of $[\varphi]$ is monomorphism into Γ, or in other words a subterminal in $\widehat{\square}/\Gamma$. The term $f : [\varphi] \to A$ corresponds to a map from $[\varphi]$ to A. As this is a subterminal we could view it as a partial element of A, or under the analogy to spaces we can think of A like a bundle over Γ, we specify an element in only some fibers of A. The a now corresponds to a full element of A that agrees with the partial element, where it is defined. So giving an element of the type $\mathrm{TFib}\, A$ is like giving an operation that can fill partial elements. Or formulated as a lifting problem:

$$\begin{tikzcd}
{[\varphi]} \arrow[r, "f"] \arrow[d, tail] & A \arrow[d] \\
\Gamma \arrow[r, "\mathrm{id}"'] \arrow[ur, dashed, "a"] & \Gamma
\end{tikzcd} \tag{6}$$

It is an opration that given $[\varphi]$ and f returns an a and a wittness that the upper triangle commutes. The lower one commuts automatically by the virtue of a being a term of type A. Notice that uniformity does not appear explicitly in this description, but happens automatically as pullback squares are the interpretation of substitutions, and the type thoeretic rules around substitution already imply a uniform choice of our lifts.

We can also give an anolog description of fibrations. As we do not need to think about equivariance, or the generic point the descriptions gets a bit less involved but stays basically the same.

4.2.3 Definition. Let $c : C \rightarrowtail \mathbb{I}^n$ be monic a *generating trivial cofibration* is a map of the form $c \mathbin{\hat{\times}} \delta_k$

4.2.4 Definition. A *uniform fibration* is a uniform right lifting map against generating trivial cofibrations.

4.2.5 Proposition ([CS22, Remark 4.2.26])**.** The uniform fibrations of $\square_{\wedge\vee}$ are equivariant.

There is an analog internal description to the trivial fibration case, just a bit more involved. For details see [OP18, Section 5], the very short version of this is, that we can describe the the classifying type directly in the internal language of the topos, having a specific form,

$$\mathrm{isFib}(A) \coloneqq (p : \mathbb{I} \to \Gamma) \to \mathrm{Comp}\,(A \circ p)$$

where Comp is some term, for the details see [OP18, Eq 5.10]. Terms of this type correspond to terms of $\mathrm{Comp}\,(A \circ p)$ in the context of $\Gamma^{\mathbb{I}}$. We can then extract the classifying types back out of it by means of the right adjoint of $(-)^{\mathbb{I}} \dashv \sqrt{-}$. Transposing

$$\begin{tikzcd} \Gamma^{\mathbb{I}} \arrow[r, dashed] \arrow[dr, "\mathrm{id}"] & \Sigma p : \Gamma^{\mathbb{I}}, \mathrm{Comp}\,(A \circ p) \arrow[d, "p_1"] \\ & \Gamma^{\mathbb{I}} \end{tikzcd}$$

to

$$\begin{tikzcd} \Gamma \arrow[r, dashed] \arrow[rr, dashed, bend left] \arrow[dr, "\mathrm{id}"] & \mathrm{Fib}(A) \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & \sqrt{\Sigma p : \Gamma^{\mathbb{I}}, \mathrm{Comp}\,(A \circ p)} \arrow[d, "\sqrt{p_1}"] \\ & \Gamma \arrow[r, "\eta"] & \sqrt{\Gamma^{\mathbb{I}}} \end{tikzcd}$$

It might look like we had won exactly nothing, but that isn't true. If we plug in for the type A over Γ our Hofmann-Streicher universe, this will directly extract us a universe for fibrations. For details on this procedure see [LOPS, Theorem 5.2].

5 Equivalence to spaces

For this section we fix the following notation. We write n for a set with cardinality n, and identify 2 with the set $\{\bot, \top\}$. We write $[n]$ for the well ordered sets $\{0, \dots, n\}$.

We would love to compare the model categories on $\widehat{\square}$ and $\widehat{\Delta}$. For this, we want at least an adjoint pair between those. If we somehow could get a functor on the site level, we could get an adjoint triple. But there is no immediate obvious functor to do the job. The usual trick would be to go look at the idempotent completions of these categories. These usually have more objects and are sometimes more suitable as the codomain of a functor, while the presheaf category stays the same (up to equivalence). For example, the idempotent completion of $\square_{\wedge\vee}$ is the category $\mathbf{FL}$ of finite lattices and monotone maps [SW21; Sat19]. So we have automatically $\widehat{\mathbf{FL}} = \widehat{\square_{\wedge\vee}}$, and there is an obvious inclusion $i : \Delta \to \mathbf{FL}$. Sadly for us, $\square$ is already idempotent complete. To get around this issue, we can embed $j : \square \to \square_{\wedge\vee} \to \mathbf{FL}$, and get into the following situation.

$$\begin{tikzcd}
\widehat{\square} \arrow[r, bend left, "j_!"] \arrow[r, bend right, "j_*"'] & \widehat{\mathbf{FL}} \arrow[l, "j^*"'] \arrow[r, "i^*"] & \widehat{\Delta} \arrow[l, bend right, "i_!"'] \arrow[l, bend left, "i_*"]
\end{tikzcd}$$

We can also define j explicitly like this:

5.0.1 Definition. Let $\mathbb{2}$ be the finite order $\bot \leq \top$. If it is convenient we will identify this with the order $0 \leq 1$.

5.0.2 Definition. Let $j : \square \to \mathbf{FL}$ be the functor given by the following data:

- On objects: $f(\{\bot|x_1...x_n|\top\}) := \mathbb{2}^n$
- On morphisms: $j(f)(g)(x) := \begin{cases} g(f(x)) & f(x) \notin \mathbb{2} \\ f(x) & f(x) \in \mathbb{2} \end{cases}$

In [Rie20], it is claimed that $i^*j_!$ is the triangulation functor and $j^*i_!$ a left Quillen homotopy inverse. In the mean time Reid Barton observed in an yet unpublished paper, that this triangulation can be described by a single functor on the level of sites $\Delta \to \square$. One could now establish Quillen equivalence using this functor, for details see [ACCRS, sec. 6]. We are going to verify the original argument. To give an overview of the necessary steps:

1. Show that $i_!$ and i^* are left Quillen.
2. Show that $j_!$ and j^* are left Quillen.

* Conclude that all 4 preserve weak equivalences.

3. Show that $i_!j^*$ and $j_!i^*$ descend to inverses of the Homotopy Categories.

Before we jump into the formal proofs we like to get some intuition for the four functors of interest. The category $\mathbf{FL}$ has the luxury of containing both Δ as the well orderings and $\square_{\wedge\vee}$ as $\mathbb{2}^n$. As $\mathbf{FL}$ is the idempotent completion of $\square_{\wedge\vee}$, we also know that a presheaf on $\mathbf{FL}$ is completely determined by its behavior on $\square_{\wedge\vee}$. The two functors i^* and j^* are just the restrictions, and we can also understand $i_!$ and $j_!$ geometrically. Let $F \in \widehat{\Delta}$, then $F(x) = i_!(F)(i(x))$. The interesting question is: what does $i_!$ define on cubes (aka $\text{よ}\mathbb{2}^n$) and $j_!$ on simplicials (aka $\text{よ}[n]$). $j_!$ will basically triangulate the cube (see Proposition 5.3.2), while $i_!$ will add just all possible cubes that are degenerated to simplicials.

To understand i^* a bit better we might take a look how things that are defined on $\mathbb{2}^n$ and extend this to simplicials. For this, we need to exhibit $[n]$ as a retract of $\mathbb{2}^n$. To do this we define two maps.

5.0.3 Definition. Let

- $r : \mathbb{2}^n \to [n]$ be the monotone function $r(f) = \Sigma_{x \in n} f(x)$
- $i : [n] \to \mathbb{2}^n$ be the monotone function $i(x)(y) = \begin{cases} \top & y < x \\ \bot & \text{otherwise} \end{cases}$

We can see directly that this is a split retract pair and thus gives rise to an idempotent $e = ir : \mathbb{2}^n \to \mathbb{2}^n$. And by general abstract nonsense we get that r is an absolute coequalizer map of e and id. Similarly i is an absolute equalizer of e and id. So for a presheaf F on **FL**, we can compute $F([n])$ as the equalizer of $F(e)$ and $F(\mathsf{id})$, or the coequalizer of $F(e)$ and $F(\mathsf{id})$ in **Set**.

5.0.4 Proposition. $i_! \Delta^n$ agrees with $\text{よ}[n]$.

Proof. We have

$$\mathrm{Hom}_{\mathbf{FL}}(i_! \Delta^n, B) = \mathrm{Hom}_{\hat{\Delta}}(\Delta^n, i^* B) = B(i([n])) = B([n]) = \mathrm{Hom}_{\mathbf{FL}}(\text{よ}[n], B),$$

and by uniqueness of adjoints, the claim follows. □

5.0.5 Proposition. $j_!(\mathbb{I}^n) = \text{よ}\mathbb{2}^n$

Proof. We have that $\mathrm{Hom}(j_!(\mathbb{I}^n), B) = \mathrm{Hom}(\mathbb{I}^n, j^*(B))$ and we get

$$\mathrm{Hom}(\text{よ}\mathbb{2}^n, B) = B(\mathbb{2}^n) = B(j(n)) = \mathrm{Hom}(\mathbb{I}^n, j^* B).$$

Thus, by uniqueness of adjoints, we have that $j_!(\mathbb{I}^n) = \text{よ}\mathbb{2}^n$. □

5.1 $i_!$ and i^* are left Quillen functors

We need to show that $i_!$ and i^* preserve cofibrations (or in other words monomorphisms) and trivial cofibrations. The hard part will be to show that $i_!$ preserves monomorphisms which we will do at the end.

5.1.1 Lemma. Let $F : \mathcal{A} \to \mathcal{B}$ be a functor, then the restriction functor $F^* : \hat{\mathcal{B}} \to \hat{\mathcal{A}}$ preserves monomorphisms.

Proof. Let f be a monomorphism in $\hat{\mathcal{B}}$. As being monic can be tested componentwise, we need to show that $(i^* f)_x$ is monic for an arbitrary $x \in \mathcal{A}$. This is just $f_{(ix)}$ which is monic by assumption. □

5.1.2 Proposition. i^* preserves monomorphisms.

Proof. Follows directly by Lemma 5.1.1. □

5.1.3 Proposition. i^* preserves trivial cofibrations.

Proof. We are in the comfortable position of being left and right adjoint. i^* preserves the interval inclusion. As pushout products with the interval inclusion sends cofibrations to trivial cofibrations and i^* preserves monomorphisms by Proposition 5.1.2, i^* sends trivial cofibrations to trivial cofibrations. □

As the next proof is mostly a straight forward induction, we don't repeat the full argument and only give a proof sketch.

5.1.4 Proposition ([Sat19, prop. 3.6]). $i_!$ preserves trivial cofibrations.

Proof sketch. It is fairly easy to see that $i_!(よ-)$ on morphisms is valued in weak equivalences, by checking that representables in $\widehat{\square_{\wedge\vee}}$ are weakly contractible. Then one shows inductively that one can build the horn inclusions as pushouts of horn inclusion of dimension n and an inclusion of the n-dimensional horn into a $n+1$ dimensional horn. As $i_!$ is cocontinuous and the trivial cofibrations have as a left lifting class the necessary closure properties, the claim follows. □

We now get to the difficult part of step 1. The argument we are going to follow is due to Sattler [Sat19] and involves the Reedy Structure of Δ. We are going through the whole argument here, as it is a good preparation for $j_!$ in step 2. We also use this to add a lot of detail and improve the criterion a bit, see the remark at Lemma 5.1.17.

5.1.5 Proposition ([Sat19, prop. 3.3]). $i_!$ preserves monomorphisms.

Following [Sat19] we are first going to prove two technical lemmas. Note that a repetition of this argument is also good to check if we actually need an elegant Reedy structure or if we can do with just an Eilenberg-Zilber category.

5.1.6 Lemma ([Sat19, lem 3.4]). Let $\mathcal{C}$ be an Eilenberg-Zilber category. Assume $\mathcal{C}$ has pullbacks along face maps. The functor $\operatorname{colim} : \hat{\mathcal{C}} \to \mathbf{Set}$ preserves monomorphisms.

5.1.7 Remark. The original source has as an extra condition that these pullbacks preserve face and degeneracy maps. We do not need this by making use of the fact that degeneracies split.

For this proof we will make some use of the equivalence between presheaves and discrete Grothendieck fibrations. This has the benefit of exhibiting our monomorphism as a Grothendieck fibration. We also can lift the Eilenberg-Zilber structure from our site, a notion that is hard to express with presheaves.

5.1.8 Lemma. Let $\mathcal{C}$ be an Eilenberg-Zilber category, and $\mathrm{p} : Q \to \mathcal{C}$ a discrete Grothendieck fibration. Then we can lift the Eilenberg-Zilber structure to Q.

The hard part of the proof will be the lifting of absolute pushouts. In general pushouts don't need to lift, as the fiber above the pushout in the base category might be empty. If we had a split pushout like in Lemma 3.1.13, we could lift the splits to solve that problem. Sadly, not all absolute pushouts are of this form, but something a bit weaker is true. The first appearance of this observation seems to be in [Par71].

5.1.9 Lemma. A commutative square

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "q"] & B \arrow[d, "m"] \\ C \arrow[r, "n"] & P \end{tikzcd}$$

is an absolute pushout diagram if and only if there exists

1. A section $u : P \to B$ of m.
2. Morphisms $r_1, \ldots, r_k : B \to A$ and $s_1, \ldots, s_k : B \to A$ for some $k \geq 1$, such that $ps_1 = \mathrm{id}_B$, $qs_i = qr_i$ for all i, $pr_i = ps_{i+1}$ for all $i < k$, and $pr_k = um$.
3. Morphisms $t_1, \ldots, t_{l+1} : C \to A$ and $v_1, \ldots, v_l : C \to A$ for some $l \geq 0$, such that $qt_1 = \mathrm{id}_C$, $pt_i = pv_i$ for all $i < l$, $qv_i = qt_{i+1}$ for all $i \leq l$, and $pt_{l+1} = un$.

or the symmetric situation where B and C are interchanged.

Proof. First let us assume we have given all the data above (and WLOG B and C are not interchanged), and let

$$\begin{tikzcd} A \arrow[r, "p"] \arrow[d, "q"] & B \arrow[d, "b"] \\ C \arrow[r, "c"] & X \end{tikzcd}$$

be a commutative square. As all this data gets preserved by any functor we only need to prove that such a diagram is already a pushout. We now need to find a $x : P \to X$ that witnesses P as a pushout. Since m splits, x will be unique automatically. We define $x \coloneqq bu$. We now need to check if it makes both evident triangles commute. We do this by chasing through all the data we have been given.

$$xm = bum = bqr_k = cqr_k = cqs_k = bps_k = bqr_{k-1} = \ldots = bps_1 = b$$

and

$$xn = bun = bpt_{l+1} = cqt_{l+1} = cqv_l = bpv_l = bpt_l = \ldots = cqt_1 = c.$$

Now let us assume the given square is an absolute pushout. Now we need to construct all the date from above. To get the desired split u, we will apply the $\mathrm{Hom}(P, -)$ functor and get a pushout square in **Set**.

$$\begin{tikzcd}
\mathrm{Hom}(P,A) \arrow[r, "p"] \arrow[d, "q"] & \mathrm{Hom}(P,B) \arrow[d, "m"] \\
\mathrm{Hom}(P,C) \arrow[r, "n"] \arrow[ur, phantom, "\ulcorner", very near end] & \mathrm{Hom}(P,P)
\end{tikzcd}$$

That means we have a surjection from $\mathrm{Hom}(P,B) + \mathrm{Hom}(P,C) \to \mathrm{Hom}(P,P)$. A preimage of id_P under this surjection is a morphism u for that either $mu = \mathrm{id}_P$ or $nu = \mathrm{id}_P$ holds. WLOG we assume the first case.

To get $r_1, \ldots, r_k$ and $s_1, \ldots, s_k$ that relate id_B and um, we look at the image under the $\mathrm{Hom}(B, -)$ functor.

$$\begin{tikzcd}
\mathrm{Hom}(B,A) \arrow[r, "p"] \arrow[d, "q"] & \mathrm{Hom}(B,B) \arrow[d, "m"] \\
\mathrm{Hom}(B,C) \arrow[r, "n"] \arrow[ur, phantom, "\ulcorner", very near end] & \mathrm{Hom}(B,P)
\end{tikzcd}$$

We know that um and id_B, are getting sent to the same map in $\hom(B,P)$, because $mum = m = m\mathrm{id}_B$. We can now construct a sequence of maps in $\hom(B,A)$ if we alternate between taking preimages of images under p and q. Lets call the preimages along p by s_i, and the preimages along q by r_i. This procedure immediately guarantees the equations

$$qs_i = qr_i \qquad\qquad pr_i = ps_{i+1}.$$

If we start the procedure with id_B in $\hom(B,B)$, we also get $ps_1 = \mathrm{id}_B$. Because id_B and um get sent to the same element along m there exists a k and a choice of r_i and s_i such that $pr_k = um$.

The maps for condition three can be constructed with the same technique after applying the $\hom(C, -)$ functor. The index shift comes from the fact that $\mathrm{id}_C \in \hom(C,C)$ and $un \in \hom(C,B)$ are in opposite corners of the resulting diagram. $\square$

To be able to lift this structure, we make sure we can lift commuting triangles first.

5.1.10 Lemma. Let $F : Q \to \mathcal{C}$ be a discrete Grothendieck fibration, $f : A \to B$ be a morphism in Q, $g : C \to F(B)$ and $h : F(A) \to C$ such that $gh = F(f)$. We can lift g and h to form a commutative diagram in Q.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, dashed, "h'"'] & & B \\ & C' \arrow[ur, dashed, "g'"'] & \end{tikzcd}$$

$$\begin{tikzcd} F(A) \arrow[rr, "F(f)"] \arrow[dr, "h"'] & & F(B) \\ & C \arrow[ur, "g"'] & \end{tikzcd}$$

5.1.11 Remark. It also suffices to just give an object B and a commutative diagram in $\mathcal{C}$.

Proof. That we can get lifts g' and h' follows directly from the definition of discrete Grothendieck fibrations. We only need to argue that the domain of h' is indeed A and that the diagram commutes. As f and $g'h'$ are lifts of $F(f) = gh$, we get $f = g'h'$ by the uniqueness of lifts. $\square$

Proof of Lemma 5.1.8. First we need to define a degree function $\mathrm{d} : \mathrm{Obj}\, Q \to \mathbb{N}$. Let d' be the degree function from $\mathcal{C}$, we then define $\mathrm{d} := \mathrm{d}'\mathrm{p}$. The next thing to do is to show that isomorphisms preserve degree, split epis lower degree and noninvertible monomorphisms increase degree. For that it suffices to show that p preserves these. Isomorphisms and split epis are preserved by every functor, so only noninvertible monomorphisms remain. We will establish this in two steps, first showing mono preservation and then reflection of inverses.

Given some monomorphism $f : A \rightarrowtail B$ in Q and two morphisms $g, h : C \to \mathrm{p}(A)$ in $\mathcal{C}$, such that $\mathrm{p}(f)h = \mathrm{p}(f)g$, we can lift g and h to $g' : C' \to A$ and $h' : C' \to A$. By Lemma 5.1.10, we have $fg' = fh'$ and thus $g' = h'$. We directly get $g = \mathrm{p}(g') = \mathrm{p}(h') = h$ and thus $\mathrm{p}(f)$ is monic.

Assume we have a map f in Q such that $\mathrm{p}(f)$ has a (one sided) inverse. By Lemma 5.1.10, the lift of that (one sided) inverse is also a (one sided) inverse of f.

This fact also helps us directly in the next step. We need to show that every morphism factors into a split epi followed by a monomorphism. We can lift a factorization by Lemma 5.1.10, and as one sided inverses are preserved by lifting, p reflects split epis. So we only need to show that monos are preserved by liftings. Let $f : A \to B$ be a morphism in Q, such that $\mathrm{p}(f)$ is monic and let $g, h : C \to A$ such that $fg = fh$. Because $p(f)$ is monic, $\mathrm{p}(g) = \mathrm{p}(h)$, and thus g and h are liftings of the same morphism. As liftings are unique $g = h$ and f is monic.

As a last step we must establish that a span of split epis has an absolute pushout. So let $f : A \to B$ and $g : A \to C$ be spilt epimorphisms in Q. As these are preserved by functors, $\mathrm{p}(f)$ and $\mathrm{p}(g)$ are split epis too. So there is an absolute pushout of $\mathrm{p}(f)$ and $\mathrm{p}(g)$. By Lemma 5.1.9, we can extend this square, by all the extra morphisms that characterize the absolute pushout. We first lift the split established by 1. in Lemma 5.1.9, to get our candidate pushout. Lifting the rest of the data and checking that the required equalities still hold is a tedious but straight forward repeated application of Lemma 5.1.8. Afterwards we can apply Lemma 5.1.9 again to verify that we indeed got an absolute pushout in Q. □

5.1.12 Remark. With the exact same argument we can lift elegant Reedy structures. We don't even need to to check the mono preservation, we can just define face maps to be lifts of face maps. Even though not explicit in the definition degeneracies in elegant reedy categories are split epimorphisms. We took the extra effort here to see if this step would be a problem in the cubical case.

5.1.13 Lemma. Let $F : Q \to \mathcal{C}$ be a discrete Grothendieck fibration. Given a span in Q and a pullback of the image of the span in $\mathcal{C}$, then there exists a lift of that pullback completing the original span to a pullback square.

Proof. Let $p : A \to C$ and $q : B \to C$ be a span in Q, such that its image under F completes to a pullback square. We can lift this square to a commuting square in Q by Lemma 5.1.10. We now need to check that this again a pullback square.

$$\begin{tikzcd}
X \arrow[dr, dashed] \arrow[rrr, bend left, "y"] \arrow[dd, bend right, "x"'] & & & \\
& P' \arrow[rr, "p'"] \arrow[dl, "q'"'] & & A \arrow[dl, "f"] \\
B \arrow[rr, "g"'] & & C &
\end{tikzcd}$$

$$\begin{tikzcd}
F(X) \arrow[dr, dashed] \arrow[rrr, bend left, "F(y)"] \arrow[dd, bend right, "F(x)"'] & & & \\
& P \arrow[rr, "p"] \arrow[dl, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & & F(A) \arrow[dl, "F(f)"] \\
F(B) \arrow[rr, "F(g)"'] & & F(C) &
\end{tikzcd}$$

So let us take any span (x, y) that completes the cospan (f, g) to a commuting square in Q. We can map this to $\mathcal{C}$ via F and get a universal map witnessing that the square in $\mathcal{C}$ is indeed a pullback. We can lift this map uniquely to P' along F, and by lemma 5.1.10 this lift makes the diagram commute. □

Proof of Lemma 5.1.6. Let $f : P' \rightarrowtail Q'$ be a monomorphism in $\hat{\mathcal{C}}$. This gives equivalently rise to a monomorphism F of discrete Grothendieck fibrations via the category of elements. We define $P \coloneqq \int P'$, $Q \coloneqq \int Q'$ and $F \coloneqq \int f$.

$$\begin{tikzcd}
P \arrow[rr, tail, "F"] \arrow[dr] & & Q \arrow[dl] \\
& \mathcal{C} &
\end{tikzcd}$$

The fact that colim sends f to a monomorphism, is equivalent to the statement that F is monic on connected components. That follows directly from the explicit construction of colimits in **Set**. Also F gets to be a discrete Grothendieck fibration on its own. We can lift the Eilenberg-Zilber structure to Q by Lemma 5.1.8.

So now let S and T be be objects of P such that $F(S)$ and $F(T)$ lie in the same connected component of Q. This means that there is a zigzag connecting $F(S)$ and $F(T)$. We can normalize this to a span. To see this, we are going to show how to turn a cospan into a span of morphisms. So let $f : D \to B$ and $g : E \to B$ be a cospan. Because of the Eilenberg-Zilber structure, we can factor those into a degeneracy followed by a face map. Lets call them f_d, f_f, g_d and g_f.

$$\begin{tikzcd}
D \arrow[dr, two heads, "f_d"] & & P \arrow[dl, tail] \arrow[dr, tail] \arrow[dd, phantom, very near start, "\lrcorner"] & & E \arrow[dl, two heads, "g_d"'] \\
& \bullet \arrow[ul, bend left, tail] \arrow[dr, tail, "f_f"] & & \bullet \arrow[ur, bend right, tail] \arrow[dl, tail, "g_f"'] & \\
& & B & &
\end{tikzcd}$$

We can then construct the pullback P along f_f and g_f, and the resulting maps concatenated with the splits of f_d and g_d are a span over D and E. This means $F(S)$ and $F(T)$ are connected by a span of morphisms. Because F is injective on objects we can lift this span to a span from S to T and thus S and T are in the same connected component. □

Before we continue to follow [Sat19], we need yet another lemma about lifting elegant Reedy structures. Sadly this does not work well for Eilenberg-Zilber categories. So we need to find a way around this later.

5.1.14 Lemma. Let $\mathcal{A}$ be an elegant Reedy category, $i : \mathcal{A} \to \mathcal{B}$ a functor and $B \in \mathcal{B}$ an object in $\mathcal{B}$. We can lift the elegant Reedy structure along $p : B \downarrow i \to \mathcal{A}$, where p is the evident projection from the comma category.

Proof. First we need to define a degree function $\mathrm{d} : \mathrm{Obj}(\mathcal{B}) \to \mathbb{N}$. Let d' be the degree function from the elegant Reedy structure. We can then define $\mathrm{d} := \mathrm{d}' p$. We define the degeneracies and faces, as being degeneracies and faces under p. Here we would already get a problem with EZ-categories as we might have maps that are monic in $B \downarrow i$, but not in $\mathcal{B}$.

As a next step we need to construct the desired factorization into a degeneracy and a face map. So we take a map f in $B \downarrow i$ and factor it as a map in $\mathcal{A}$ into $f = f_f f_d$. If we try to lift the factorization we end up with the following diagram:

$$\begin{tikzcd}
& B \arrow[dl, "\phi"'] \arrow[dr, "\phi'"] \arrow[dd, "i(f_d)\phi"] & \\
i(A) \arrow[rr, "i(f)"] \arrow[dr, "i(f_d)"'] & & i(A') \\
& i(X) \arrow[ur, "i(f_f)"'] &
\end{tikzcd}$$

It remains to show that the front right triangle commutes.

$$i(f_f)(i(f_d)\phi) = (i(f_f)i(f_d))\phi = i(f_f f_d)\phi = i(f)\phi = \phi'$$

It is then tedious, but straight forward to check one of the equivalent elegance axioms. □

5.1.15 Lemma. Let $i : \mathcal{A} \to \mathcal{B}$ be a functor, and B an object of $\mathcal{B}$, and let $p : B \downarrow i \to A$ be the projection map. A span in $B \downarrow i$ has a pullback, if the span has a pullback in $\mathcal{B}$, and the pullback square lies in the image of i.

5.1.16 Remark. Usually this situation is given, if $\mathcal{A}$ has pullbacks along a class of maps and i preserves them.

Proof. So consider a span of maps in $B \downarrow i$, that fulfills the conditions of this lemma. If we unravel this, we get the following diagram:

$$\begin{tikzcd} & B \arrow[dl, bend right, "\phi"'] \arrow[dr, bend left, "\phi'"] & \\ i(X) \arrow[dr, "i(f)"] & & i(Y) \arrow[dl, "i(g)"'] \\ & i(Z) & \end{tikzcd}$$

By assumption we have pullbacks of $i(f)$ and $i(g)$ in $\mathcal{B}$ that lie in the image of i. So we can complete the diagram

$$\begin{tikzcd} & B \arrow[d, dashed, "\langle \phi, \phi' \rangle"] \arrow[ddl, bend right, "\phi"'] \arrow[ddr, bend left, "\phi'"] & \\ & i(P) \arrow[dl] \arrow[dr] & \\ i(X) \arrow[dr, "i(f)"] & & i(Y) \arrow[dl, "i(g)"'] \\ & i(Z) & \end{tikzcd}$$

and exhibit $i(P)$ as an object in $B \downarrow i$ by the universal map of the pullback. □

5.1.17 Lemma ([Sat19, lemma 3.5]). Let $\mathcal{A}$ be an elegant Reedy category and $i : \mathcal{A} \to \mathcal{B}$ a functor. Assume that $\mathcal{A}$ has pullbacks along face maps whenever the cospan under consideration lies in the image of the projection $B \downarrow i \to \mathcal{A}$ for some $B \in \mathcal{B}$, and that i preserves these pullbacks. Then the left Kan extension $i_! : \hat{\mathcal{A}} \to \hat{\mathcal{B}}$ preserves monomorphisms.

5.1.18 Remark. The original source also requires that these pullbacks preserve face and degeneracy maps. We get around this by making use of the fact that degeneracies are split in Lemma 5.1.8.

Proof. We only need to show that $i_!$ preserves monomorphisms componentwise, giving us a functor from $\hat{\mathcal{A}} \to \mathbf{Set}$ for every object in $B \in \mathcal{B}$. This functor can be written as

$$\hat{\mathcal{A}} \xrightarrow{p^*} \widehat{B \downarrow i} \xrightarrow{\text{colim}} \mathbf{Set}$$

where $p : B \downarrow i \to \mathcal{A}$ is the projection. This is the usual construction of $i_!$ in this setting, see for example [Sta]. It is trivial that p^* preserves monomorphisms, so we only need to check that colim does. We want to apply Lemma 5.1.6. We can lift the elegant Reedy structure from $\mathcal{A}$ to $B \downarrow i$ along p by Lemma 5.1.14. And as the face maps in $B \downarrow i$ are defined as maps that p sends to face maps in $\mathcal{A}$. We can lift the required pullbacks to $B \downarrow i$ by Lemma 5.1.15. □

Proof of Proposition 5.1.5. We want to verify the conditions for Lemma 5.1.17. Our first observation is that Δ does not have arbitrary pullbacks of face maps. Intuitively speaking we have only the non-empty ones. So let us unpack the condition that we only need pullbacks if the cospan of face maps lies in the image of the projection $B \downarrow i \to \Delta$ for some B. That means there is a $B \in \mathbf{FL}$ such that the following square commutes:

$$\begin{tikzcd} & B \arrow[dl] \arrow[dr] & \\ i(X) \arrow[dr, "i(f_f)"'] & & i(Y) \arrow[dl, "i(g_f)"] \\ & i(Z) & \end{tikzcd}$$

As the face maps in Δ are monic, we can identify X and Y with their respective image in Z. As $\mathbf{FL}$ does not contain the empty poset, they share at least a point, and the pullback is given by $\operatorname{im} f_f \cap \operatorname{im} g_f$. A quick calculation shows that this is also a pullback in $\mathbf{FL}$ and thus we can apply Lemma 5.1.17 □

5.2 j^* and $j_!$ are left Quillen

We are now going to show that j^* and $j_!$ are left Quillen. As expected, the hard part will be again to show that $j_!$ preserves monomorphisms.

So let us start with j^* again.

5.2.1 Proposition. j^* preserves monomorphisms.

Proof. Follows directly from Lemma 5.1.1 □

5.2.2 Proposition. j^* preserves trivial cofibrations.

Proof. Like in Proposition 5.1.3, we are in the comfortable position of being left and right adjoint. Note that j^* also preserves the interval inclusion. As pushout products with the interval inclusion send cofibrations to trivial cofibrations and j^* preserves monomorphisms by Proposition 5.2.1, j^* sends trivial cofibrations to trivial cofibrations. □

5.2.3 Proposition. j^* preserves fibrations

Proof. Preservation of uniform fibrations is clear, as every lifting problem against a generating trivial cofibration factors throug the image of a trivial cofibration under j^*. As j^* preserves coequalizer the equivariance follows from the equivariance of the model category on $\widehat{\square_{\wedge\vee}}$, which follows from the presence of (at least one) connection. $\square$

We want to recap the general proof idea that $i_!$ preserved monos. We can test being a monomorphism component wise and can write the left Kan extension evaluated at a point as some mono preserving functor followed by a colimit. That the colimit preserves monomorphisms is equivalent to a condition of connected components on the categories of elements. We can only lift morphisms backwards along discrete Grothendieck fibrations, but we are faced with zigzags. To remedy this we want to lift a strong enough pullback property to turn these zigzags into spans.

The strategy will be the same for $j_!$. Sadly we couldn't get arbitrary Eilenberg-Zilber structures to lift, so we will need a slightly different point of attack.

We first observe that $\square$ has a lot more pullbacks then Δ. While Δ has only "non-empty" pullbacks of face maps, $\square$ has all "non-empty" pullbacks. This elevates us from the necessity to lift the Eilenberg-Zilber structure to $B \downarrow i$. We can get by with the following lemma.

5.2.4 Lemma. Let $i : \mathcal{A} \to \mathcal{B}$ be a functor such that for all $B \in \mathcal{B}$, spans in the image of the projection $p : B \downarrow i$ have pullbacks in $\mathcal{B}$, and these pullbacks are in the image of i. Then $i_!$ preserves monomorphisms.

To show this we replace Lemma 5.1.6 with the following simpler lemma.

5.2.5 Lemma. Let $\mathcal{C}$ be a category. Assume $\mathcal{C}$ has pullbacks, then the functor $\operatorname{colim} : \hat{\mathcal{C}} \to \mathbf{Set}$ preserves monomorphisms.

Proof. Again let f be a monomorphism in $\hat{\mathcal{C}}$. This is equivalently a monomorphism $F : P \to Q$ of discrete Grothendieck fibrations over $\mathcal{C}$. We need to show that F is injective on connected components. We can lift the pullbacks to Q by Lemma 5.1.13. Let now S and T be in Q such that $F(S)$ and $F(T)$ are connected by a zigzag. We can turn this zigzag into a span, by taking pullbacks of cospans. Because F is monic it is injective on objects and we can lift this span to a span in P and thus S and T are in the same connected component. $\square$

Proof of Lemma 5.2.4. As in Lemma 5.1.17, we check that $i_!$ preserves monos component wise. We write $i_!(B)$ again as

$$\hat{\mathcal{A}} \xrightarrow{p^*} \widehat{B \downarrow i} \xrightarrow{\operatorname{colim}} \mathbf{Set}$$

where $p : B \downarrow i \to A$ is the projection map. We need to show that $B \downarrow i$ has pullbacks, but that is an immediate consequence of Lemma 5.1.15. $\square$

Finally we arrive at the conclusion we wanted to get to:

5.2.6 Proposition. $j_! : \widehat{\mathbf{FL}} \to \widehat{\square}$ preserves monomorphisms.

Proof. For this proof we will identify $2 = \{\bot, \top\}$. We want to fulfill the conditions of Lemma 5.2.4. To check these conditions, we check that equalizers and products with the desired conditions exist. Products are a straightforward calculation so we go forward to the equalizer case. To check this we will look at a diagram of the following form,

$$\begin{tikzcd} & B \arrow[dl] \arrow[dr] & \\ \mathbb{2}^n \arrow[rr, shift left, "j(f)"] \arrow[rr, shift right, "j(g)"'] & & \mathbb{2}^m \end{tikzcd}$$

where we identify f and g with functions $m \to n + 2$. We note that equalizers, if they exist in **FL**, are constructed the same way as in **Set**. If we apply this to our diagram, we can see that this construction can't create something empty, as both maps need to preserve elements from B. We would like to complete the diagram in the following way

$$\begin{tikzcd} & B \arrow[dl] \arrow[d] \arrow[dr] & \\ \mathbb{2}^l \arrow[r, "j(e)"] & \mathbb{2}^n \arrow[r, shift left, "j(f)"] \arrow[r, shift right, "j(g)"'] & \mathbb{2}^m \end{tikzcd}$$

So we need to find a finite set l and a map $e : n \to l + 2$. We will construct those by effectively constructing the coequalizer in $\mathbf{FinSet}^*_*$. So we take the quotient ${n+2}/{\sim}$ where $\sim$ is the equivalence relation generated by the relation of having a joint preimage under f and g. We will now argue that our equalizer would be empty if this quotient identifies $\top$ and $\bot$.

We assume that indeed the relation identifies $\bot$ and $\top$. Let $h \in \mathbb{2}^n$ such that $j(f)(h) = j(g)(h)$. Because $\bot$ and $\top$ are being identified, there exists two finite sequence of elements x_i and y_i in m, such that there exists a k and (WLOG) $f(x_1) = \bot$, $g(x_i) = f(y_i)$, $g(y_i) = f(x_{i+1})$ and $g(y_k) = \top$. Plugging that into the definition of j, we get

$$\begin{aligned} \bot = j(f)(h)(x_1) &= j(g)(h)(x_1) = j(f)(h)(y_1) \\ &= j(g)(h)(y_1) = j(f)(h)(x_2) = \dots = j(g)(h)(y_k) = \top \end{aligned}$$

which is a contradiction and thus $\top$ and $\bot$ are not identified. For $e : n \to l+2$, we take the evident quotient map restricted to n. While we now have our candidate, we still need to show that this is actually a pushout in **FL**. As **FL** is well pointed it suffices to check global elements, which again are just elements in the set theoretic sense. As $fe = ge$ by construction and j is a functor we get $j(f)j(e) = j(g)j(e)$.

Again let $h \in \mathbb{2}^n$ such that $j(f)(h) = j(g)(h)$. We can then define

$$h' : l \to \mathbb{2} \qquad \text{by} \qquad h'(x) := h(z) \qquad \text{where} \qquad z \in e^{-1}(x).$$

For this definition to make sense we need to argue that the choice of z does not matter. So let z and z' be both element of $e^{-1}(x)$. Like above, that means there are two sequences x_i and y_i, just that $f(x_1) = z$ and $g(y_k) = z'$, instead of $\bot$ and $\top$. And like above we get

$$\begin{aligned} h(z) = j(f)(h)(x_1) &= j(g)(h)(x_1) = j(f)(h)(y_1) \\ &= j(g)(h)(y_1) = j(f)(h)(x_2) = \dots = j(g)(h)(y_k) = h(z') \end{aligned}$$

We also need that the map $j(e)$ is monic, but this follows directly from the fact that e in $\mathbf{FinSet}^*_*$ is epic. We only need to produce a map $B \to \mathbb{2}^l$, that commutes. We get this map directly, because we have shown that this diagram is an equalizer in **FL**. This means we fulfill the conditions of Lemma 5.2.4 and thus $j_!$ preserves monomorphisms. $\square$

5.3 $i^* j_!$ and $j^* i_!$ induce an equivalence of homotopy categories

Before we continue our plan directly, we take a short detour to get a little bit better idea how these two functors behave.

From the previous sections it is immediately clear that $i^* j_!$ and $j^* i_!$ are left Quillen functors with some right adjoint. One might ask if these two functors are adjoint to each other. The former development would suggest that they aren't and that is indeed the case.

5.3.1 Definition. Let $t : \square \to \widehat{\Delta}$ be the functor that sends $[1]^n \mapsto (\Delta^1)^n$. The *triangulation functor* T is the left Kan extension of t along Yoneda.

$$\begin{tikzcd} \square \arrow[r, "t"] \arrow[d, "よ"'] & \widehat{\Delta} \\ \widehat{\square} \arrow[ur, "\mathrm{T}"] & \end{tikzcd}$$

5.3.2 Proposition. The functor $i^* j_!$ is the triangulation functor.

Proof. As T is the unique cocontinuous functor that extends the product preserving functor from $\widehat{\square}$ to $\widehat{\Delta}$ that sends the interval to the interval, we only need to show these conditions. $i^* j_!$ is cocontinuous as it is the composition of two left adjoints. We need to show that this functor is product preserving on representables. But by Yoneda and Proposition 5.0.5 we have

$$j_!(\mathbb{I}^n \times \mathbb{I}^m) = j_!(\mathbb{I}^{n+m}) = \text{よ}\mathbb{2}^{n+m} = \text{よ}\mathbb{2}^n \times \text{よ}\mathbb{2}^m = j_!(\mathbb{I}^n) \times j_!(\mathbb{I}^m)$$

$j_!$ preserves products of representables. Because i^* is a right adjoint we get this property immediately. We also need to show that $i^* j_!$ preserves the interval. We already know this for $j_!$. So the question is if i^* preserves the interval. We now that $i^*(\mathbb{I})(x) = \mathrm{Hom}_{\square_{\wedge\vee}}(\text{よ}[1], i(x))$. As i is fully faithful the claim follows. □

5.3.3 Example.

$$\mathrm{Hom}_{\widehat{\square}}(j^* i_!(\Delta^2), \mathbb{I}^2) \neq \mathrm{Hom}_{\widehat{\Delta}}(\Delta^2, i^* j_!(\mathbb{I}^2))$$

Intuitively the reason for this is that we can't map a square with one side degenerated to a triangle into the representable square in $\widehat{\square}$. But we have a lot of ways to map a triangle into a triangulated square in $\widehat{\Delta}$.

Proof. By the Yoneda lemma, and that $j^* i_!$ is the triangulation functor, a map on the right-hand side corresponds to a pair of monotone maps $[2] \to [1]$. There are 16 of such pairs. On the left we can use Proposition 5.0.5, and expanding the definition we get $\mathrm{Hom}_{\widehat{\square}}(\mathrm{Hom}_{\mathbf{FL}}(j(-), [2]), \mathrm{Hom}_{\square}(-, 2))$. Lets take such a transformation and call it η. η_0 gives rise to a map $f : [2] \to \mathbb{2}^2$. η_1 witnesses that f is indeed monotone, by composing with face maps. And η_2 rules out all injective maps, as for them always exactly 2 faces agree and $\mathbb{I}^2(2)$ does not contain such faces. There are only 9 such maps, which is an upper bound for the possible number of natural transformations. □

5.3.4 Example.

$$\mathrm{Hom}_{\widehat{\Delta}}(i^* j_!(\mathbb{I}^2), \Delta^1 \times \Delta^1) \neq \mathrm{Hom}_{\widehat{\square}}(\mathbb{I}^2, j^* i_!(\Delta^1 \times \Delta^1))$$

Here the problem is that the square made from a product in $\widehat{\Delta}$ doesn't have a 2-cell that fills the whole square, this doesn't give us a way to map the representable square on it without degenerating something. While if we triangulate the representable square first we don't run into that problem.

Proof. As $j^* i_!$ is the triangulation functor, the left-hand side becomes $\mathrm{Hom}_{\hat{\Delta}}(\Delta^1 \times \Delta^1, \Delta^1 \times \Delta^1)$. We can write $\Delta^1 \times \Delta^1$ as a pushout of two copies of Δ^1 joint by an edge. As $i_!$ is cocontinuous, we can construct this pushout in **FL**. By this and Proposition 5.0.4, we get a colimit of two representables there. Unraveling these definitions it is a straightforward exercise to verify that the set of on the left-hand side has more elements. □

From here on we follow the argument from [ACCRS, §6.2]. They lay down a criterion on functors and model categories such that every natural transformation between them is a weak equivalence. Thus we only need to construct any natural transformation between the identity functor and the concatenation of these functors.

For this, we cite two technical lemmas whose theory is mostly developed in [RV14], but are taken in this form from [ACCRS]. To make sense of the first lemma we need to introduce a technicality. If we have a category $\mathcal{A}$ and presheaves on it, then the automorphism group of some $a \in \mathcal{A}$ acts on $\text{よ}a\, b$ by composition. This kind of acting is natural in the sense that $\text{よ}a f$ is an $\mathrm{Aut}(a)$-equivariant map. Thus we can say $\mathrm{Aut}(a)$ acts on $\text{よ}a$. And thus we can also quotient $\text{よ}a$ by subgroups of $\mathrm{Aut}(a)$. This is also where equivariance comes into play again, as equivariance by definition guarantees us that these quotients of representables by some automorphism subgroup are contractible.

5.3.5 Lemma ([RV14, §5; ACCRS, lemma 6.2.13]). Let $\mathcal{A}$ be an Eilenberg-Zilber category. Then the monomorphisms in $\hat{\mathcal{A}}$ are generated under coproduct, pushout, sequential composition, and right cancellation under monomorphisms by the maps $\varnothing \to {}^{\text{よ}a}/_{H}$ valued in the quotient of a representable presheaf at some $a \in \mathcal{A}$ by an arbitrary subgroup H of its automorphism group.

5.3.6 Lemma ([RV14, §5; ACCRS, lemma 6.2.15]). Let $U, V : K \to M$ be a cocontinuous pair of functors valued in a model category and $\alpha : U \Rightarrow V$ a natural transformation between them. Define the cofibrations in K to be the maps that are sent to cofibrations under both U and V. Define $\mathcal{N}$ to be the class of cofibrations between cofibrant objects that are sent by Leibniz pushout application with α to weak equivalences in M. Then $\mathcal{N}$ is closed under coproducts, pushout, (transfinite) composition, and right cancellation between cofibrations.

5.3.7 Corollary ([ACCRS, corollary 6.2.16]). Let $\mathcal{A}$ be an Eilenberg–Zilber category and consider a parallel pair of functors $U, V : \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}} \to M$ valued in a model category M together with a natural transformation $\alpha : U \Rightarrow V$. Suppose that U and V preserve colimits and send monomorphisms in K to cofibrations in M. Then if the components of α at quotients of representables by subgroups of their automorphism groups are weak equivalences, then all components of α are weak equivalences.

Proof. By Lemma 2.2.12 we can write the the part of α at X as a Leibniz application with the monomorphism $\varnothing \to X$. We can use both lemmas from above to show that α gets send by all of those maps to a weak equivalence. By Lemma 5.3.5 the monomorphisms in $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ are generated by morphisms $\varnothing \to {}^{\text{よ}a}\!/_{H}$ under some closure properties. By assumption, morphisms of that form are sent to weak equivalences by the Leibniz application with α. By Lemma 5.3.6, the maps that get send to weak equivalences by Leibniz application with α have the same closure properties, thus all monomorphisms get send to weak equivalences, and with them all morphisms of the form $\varnothing \to X$. $\square$

5.3.8 Corollary ([ACCRS, Corollary 6.2.17]). Let $\mathcal{A}$ be an Eilenberg–Zilber category for which $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ admits a model structure whose cofibrations are the monomorphisms, in which the quotients ${}^{\text{よ}a}\!/_{H}$ of representables by subgroups of their automorphism groups are weakly contractible. Then if $U, V : \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}} \to M$ define a pair of left Quillen functors that preserve the terminal object, then any natural transformation $\alpha : U \Rightarrow V$ is a natural weak equivalence.

Proof. By Ken Browns lemma, left Quillen functors preserve weak equivalences between cofibrant objects. If such a functor preserves the terminal object, then it also preserves weak contractibility between cofibrant objects. To apply Corollary 5.3.7 we need show that $\alpha_{{}^{\text{よ}a}\!/_{H}}$ is a weak equivalence. As this is weakly contractible, we get the following commuting square.

$$\begin{tikzcd}
U\left({}^{\text{よ}a}\!/_{H}\right) \arrow[r, "\alpha_{{}^{\text{よ}a}\!/_{H}}"] \arrow[d, "\wr"] & V\left({}^{\text{よ}a}\!/_{H}\right) \arrow[d, "\wr"] \\
U(*) \arrow[r, equal] & V(*)
\end{tikzcd}$$

And by 2-out-of-3, the upper map must be a weak equivalence, and thus by Corollary 5.3.7 the claim follows. $\square$

5.3.9 Theorem. $j^* i_!$ and $i^* j_!$ induce an equivalence between $\mathrm{Ho}(\widehat{\square})$ and $\mathrm{Ho}(\widehat{\Delta})$.

Proof. By Propositions 5.1.2 to 5.1.5, 5.2.1 to 5.2.3 and 5.2.6, are i^*,$i_!$,j^*, and $j_!$ left Quillen functors, and thus $j^* i_!$ and $i^* j_!$ too. By Ken Browns lemma we know that left Quillen functors preserve weak equivalences between cofibrant objects.

In the model structures on $\widehat{\square}$ and $\widehat{\Delta}$, every monomorphism a cofibration and thus every object cofibrant. Thus $j^*i_!$ and $i^*j_!$ preserve weak equivalences and descend to functors between the corresponding homotopy categories.

To show that these induce an equivalence between the homotopy categories we must show that we have a zigzag of natural transformations between $i^*j_!j^*i_!$ and the identity functor such that every natural transformation is valued in weak equivalences, and likewise for $j^*i_!i^*j_!$. We will denote η^i and ε^i the unit and counit of the adjunction $i_! \dashv i^*$. Likewise we denote η^j and ε^j for the adjunction $j_! \dashv j^*$. We can construct these as follows

$$\begin{tikzcd}
\widehat{\Delta} \arrow[r, "i_!"] \arrow[d, phantom, "\Downarrow \mathrm{id}"] & \widehat{\square_{\wedge\vee}} \arrow[r, "j^*"] \arrow[d, phantom, "\circledast"] & \widehat{\square} \arrow[r, "j_!"] \arrow[d, phantom, "\Downarrow \varepsilon^j"] & \widehat{\square_{\wedge\vee}} \arrow[r, "i^*"] \arrow[d, phantom, "\circledast"] & \widehat{\Delta} \arrow[d, phantom, "\Downarrow \mathrm{id}"] \\
\widehat{\Delta} \arrow[r, "i_!"] & \widehat{\square_{\wedge\vee}} \arrow[rr, equal, "\mathrm{id}"] & \arrow[d, phantom, "\Uparrow \eta^i"] & \widehat{\square_{\wedge\vee}} \arrow[r, "i^*"] & \widehat{\Delta} \\
\widehat{\Delta} \arrow[rrrr, equal, "\mathrm{id}"] & & & & \widehat{\Delta}
\end{tikzcd}$$

and

$$\begin{tikzcd}
\widehat{\square} \arrow[r, "j_!"] \arrow[d, phantom, "\Downarrow \mathrm{id}"] & \widehat{\square_{\wedge\vee}} \arrow[r, "i^*"] \arrow[d, phantom, "\circledast"] & \widehat{\Delta} \arrow[r, "i_!"] \arrow[d, phantom, "\Downarrow \varepsilon^i"] & \widehat{\square_{\wedge\vee}} \arrow[r, "j^*"] \arrow[d, phantom, "\circledast"] & \widehat{\square} \arrow[d, phantom, "\Downarrow \mathrm{id}"] \\
\widehat{\square} \arrow[r, "j_!"] & \widehat{\square_{\wedge\vee}} \arrow[rr, equal, "\mathrm{id}"] & \arrow[d, phantom, "\Uparrow \eta^j"] & \widehat{\square_{\wedge\vee}} \arrow[r, "j^*"] & \widehat{\square} \\
\widehat{\square} \arrow[rrrr, equal, "\mathrm{id}"] & & & & \widehat{\square}
\end{tikzcd}$$

where $\circledast$ is the Godement product. By Propositions 5.0.4 and 5.0.5, $i_!$ and $j_!$ preserve terminal obejcts and i^* and j^* do so because they are right adjoints. So by Corollary 5.3.8 these two cospans of natural transformations are weak equivalences and thus we have our desired equivalence of homotopy categories. $\square$

References

[ACCRS] Steve Awodey et al. *The Equivariant Model Structure on Cartesian Cubical Sets*. June 26, 2024. DOI: 10.48550/arXiv.2406.18497. arXiv: 2406.18497 [cs, math]. URL: http://arxiv.org/abs/2406.18497 (visited on 07/03/2024). Pre-published.

[AGH21] S. Awodey, N. Gambino, and S. Hazratpour. *Kripke-Joyal Forcing for Type Theory and Uniform Fibrations*. Oct. 27, 2021. DOI: 10.48550/arXiv.2110.14576. arXiv: 2110.14576 [math]. URL: http://arxiv.org/abs/2110.14576 (visited on 02/21/2024). Pre-published.

[AHH18] Carlo Angiuli, Kuen-Bang Hou (Favonia), and Robert Harper. "Cartesian Cubical Computational Type Theory: Constructive Reasoning with Paths and Equalities." In: *DROPS-IDN/v2/Document/10.4230/LIPIcs.CSL.2018.6*. 27th EACSL Annual Conference on Computer Science Logic (CSL 2018). Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2018. DOI: 10.4230/LIPIcs.CSL.2018.6. URL: https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.CSL.2018.6 (visited on 08/09/2024).

[Ang+21] Carlo Angiuli et al. "Syntax and Models of Cartesian Cubical Type Theory." In: *Mathematical Structures in Computer Science* 31.4 (Apr. 2021), pp. 424–468. ISSN: 0960-1295, 1469-8072. DOI: 10.1017/S0960129521000347. URL: https://www.cambridge.org/core/journals/mathematical-structures-in-computer-science/article/syntax-and-models-of-cartesian-cubical-type-theory/01B9E98DF997F0861E4BA13A34B72A7D (visited on 04/21/2023).

[Awo18] Steve Awodey. "A Cubical Model of Homotopy Type Theory." In: *Annals of Pure and Applied Logic*. Logic Colloquium 2015 169.12 (Dec. 1, 2018), pp. 1270–1294. ISSN: 0168-0072. DOI: 10.1016/j.apal.2018.08.002. URL: https://www.sciencedirect.com/science/article/pii/S0168007218300861 (visited on 08/09/2024).

[Awo23a] Steve Awodey. *Cartesian Cubical Model Categories*. July 14, 2023. DOI: 10.48550/arXiv.2305.00893. arXiv: 2305.00893 [math]. URL: http://arxiv.org/abs/2305.00893 (visited on 08/03/2024). Pre-published.

[Awo23b] Steve Awodey. *On Hofmann-Streicher Universes*. July 11, 2023. DOI: 10.48550/arXiv.2205.10917. arXiv: 2205.10917 [math]. URL: http://arxiv.org/abs/2205.10917 (visited on 07/05/2024). Pre-published.

[Bar19] Reid William Barton. "A Model 2-Category of Enriched Combinatorial Premodel Categories." In: (Sept. 10, 2019). ISSN: 4201-3127. URL: https://dash.harvard.edu/handle/1/42013127 (visited on 04/28/2024).

[BCH14] Marc Bezem, Thierry Coquand, and Simon Huber. "A Model of Type Theory in Cubical Sets." In: Schloss Dagstuhl - Leibniz-Zentrum fuer Informatik GmbH, Wadern/Saarbruecken, Germany, 2014, 22 pages. DOI: 10.4230/LIPICS.TYPES.2013.107. URL: http://drops.dagstuhl.de/opus/volltexte/2014/4628/ (visited on 08/28/2023).

[BCP15] Marc Bezem, Thierry Coquand, and Erik Parmann. "Non-Constructivity in Kan Simplicial Sets." In: *DROPS-IDN/v2/Document/10.4230/LIPIcs.TLCA.2015.92*. 13th International Conference on Typed Lambda Calculi and Applications (TLCA 2015). Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2015. DOI: 10.4230/LIPIcs.TLCA.2015.92. URL: https://drops.dagstuhl.de/entities/document/10.4230/LIPIcs.TLCA.2015.92 (visited on 08/09/2024).

[BG16a] John Bourke and Richard Garner. "Algebraic Weak Factorisation Systems I: Accessible AWFS." In: *Journal of Pure and Applied Algebra* 220.1 (Jan. 2016), pp. 108–147. ISSN: 00224049. DOI: 10.1016/j.jpaa.2015.06.002. arXiv: 1412.6559. URL: http://arxiv.org/abs/1412.6559 (visited on 04/26/2022).

[BG16b] John Bourke and Richard Garner. "Algebraic Weak Factorisation Systems II: Categories of Weak Maps." In: *Journal of Pure and Applied Algebra* 220.1 (Jan. 2016), pp. 148–174. ISSN: 00224049. DOI: 10.1016/j.jpaa.2015.06.003. arXiv: 1412.6560. URL: http://arxiv.org/abs/1412.6560 (visited on 04/26/2022).

[Cam23] Timothy Campion. *Cubical Sites as Eilenberg-Zilber Categories*. Mar. 10, 2023. DOI: 10.48550/arXiv.2303.06206. arXiv: 2303.06206 [math]. URL: http://arxiv.org/abs/2303.06206 (visited on 08/09/2024). Pre-published.

[CCHM] Cyril Cohen et al. *Cubical Type Theory: A Constructive Interpretation of the Univalence Axiom*. Nov. 7, 2016. DOI: 10.48550/arXiv.1611.02108. arXiv: 1611.02108 [cs, math]. URL: http://arxiv.org/abs/1611.02108 (visited on 03/29/2024). Pre-published.

[CHM18] Thierry Coquand, Simon Huber, and Anders Mörtberg. "On Higher Inductive Types in Cubical Type Theory." In: *Proceedings of the 33rd Annual ACM/IEEE Symposium on Logic in Computer Science*. LICS '18. New York, NY, USA: Association for Computing Machinery, July 9, 2018, pp. 255–264. ISBN: 978-1-4503-5583-4. DOI: 10.1145/3209108.3209197. URL: https://dl.acm.org/doi/10.1145/3209108.3209197 (visited on 08/09/2024).

[CMS16] Evan Cavallo, Anders Mörtberg, and Andrew W Swan. "Unifying Cubical Models of Univalent Type Theory." In: (2016).

[Coq14] Thierry Coquand. *Variation on Cubical Sets*. 2014. URL: https://www.cse.chalmers.se/~coquand/diag.pdf (visited on 04/30/2024).

[CS22] Evan Cavallo and Christian Sattler. *Relative Elegance and Cartesian Cubes with One Connection*. Nov. 27, 2022. DOI: 10.48550/arXiv.2211.14801. arXiv: 2211.14801 [math]. URL: http://arxiv.org/abs/2211.14801 (visited on 12/16/2022). Pre-published.

[Gar09] Richard Garner. "Understanding the Small Object Argument." In: *Applied Categorical Structures* 17.3 (June 2009), pp. 247–285. ISSN: 0927-2852, 1572-9095. DOI: 10.1007/s10485-008-9137-4. arXiv: 0712.0724. URL: http://arxiv.org/abs/0712.0724 (visited on 04/26/2022).

[GK13] Nicola Gambino and Joachim Kock. "Polynomial Functors and Polynomial Monads." In: *Mathematical Proceedings of the Cambridge Philosophical Society* 154.1 (Jan. 2013), pp. 153–192. ISSN: 0305-0041, 1469-8064. DOI: 10.1017/S0305004112000394. arXiv: 0906.4931 [math]. URL: http://arxiv.org/abs/0906.4931 (visited on 08/04/2024).

[GS17] Nicola Gambino and Christian Sattler. "The Frobenius Condition, Right Properness, and Uniform Fibrations." In: *Journal of Pure and Applied Algebra* 221.12 (Dec. 2017), pp. 3027–3068. ISSN: 00224049. DOI: 10.1016/j.jpaa.2017.02.013. arXiv: 1510.00669. URL: http://arxiv.org/abs/1510.00669 (visited on 04/29/2022).

[Hov07] Mark Hovey. *Model Categories*. American Mathematical Society. Oct. 17, 2007. DOI: 10.1090/surv/063. URL: https://www.ams.org/surv/063 (visited on 04/28/2024).

[HR24] Sina Hazratpour and Emily Riehl. *A 2-Categorical Proof of Frobenius for Fibrations Defined from a Generic Point*. Feb. 22, 2024. DOI: 10.48550/arXiv.2210.00078. arXiv: 2210.00078 [math]. URL: http://arxiv.org/abs/2210.00078 (visited on 08/08/2024). Pre-published.

[HS97] Martin Hofmann and Thomas Streicher. "Lifting Grothendieck Universes." In: (Spr. 1997). URL: https://www2.mathematik.tu-darmstadt.de/~streicher/NOTES/lift.pdf.

[KL21] Krzysztof Kapulkin and Peter LeFanu Lumsdaine. "The Simplicial Model of Univalent Foundations (after Voevodsky)." In: *Journal of the European Mathematical Society* 23.6 (Mar. 8, 2021), pp. 2071–2126. ISSN: 1435-9855. DOI: 10.4171/jems/1050. URL: https://ems.press/journals/jems/articles/274693 (visited on 08/09/2024).

[LOPS] Daniel R. Licata et al. "Internal Universes in Models of Homotopy Type Theory." 2018. DOI: 10.4230/LIPIcs.FSCD.2018.22. arXiv: 1801.07664 [cs]. URL: http://arxiv.org/abs/1801.07664 (visited on 05/11/2022).

[LS05] Stephen Lack and Paweł Sobociński. "Adhesive and Quasiadhesive Categories." In: *RAIRO - Theoretical Informatics and Applications - Informatique Théorique et Applications* 39.3 (2005), pp. 511–545. ISSN: 1290-385X. DOI: 10.1051/ita:2005028. URL: http://www.numdam.org/item/ITA_2005__39_3_511_0/ (visited on 08/07/2024).

[Mail] *Quillen Model Structure*. URL: https://groups.google.com/g/homotopytypetheory/c/RQkLWZ_83kQ (visited on 08/09/2024).

[OP18] Ian Orton and Andrew M. Pitts. "Axioms for Modelling Cubical Type Theory in a Topos." Dec. 8, 2018. DOI: 10.23638/LMCS-14(4:23)2018. arXiv: 1712.04864 [cs]. URL: http://arxiv.org/abs/1712.04864 (visited on 05/11/2022).

[Par71] Robert Paré. "On Absolute Colimits." In: *Journal of Algebra* 19.1 (Sept. 1, 1971), pp. 80–95. ISSN: 0021-8693. DOI: 10.1016/0021-8693(71)90116-5. URL: https://www.sciencedirect.com/science/article/pii/0021869371901165 (visited on 07/10/2024).

[Rie11] Emily Riehl. “Algebraic Model Structures.” Mar. 11, 2011. arXiv: 0910.2733 [math]. URL: http://arxiv.org/abs/0910.2733 (visited on 04/26/2022).

[Rie20] Emily Riehl, director. *The Equivariant Uniform Kan Fibration Model of Cubical Homotopy Type Theory - YouTube*. May 26, 2020. URL: https://www.youtube.com/watch?v=A8tEwxM7uxE (visited on 06/19/2022).

[RV14] Emily Riehl and Dominic Verity. *The Theory and Practice of Reedy Categories*. June 3, 2014. DOI: 10.48550/arXiv.1304.6871. arXiv: 1304.6871 [math]. URL: http://arxiv.org/abs/1304.6871 (visited on 04/07/2024). Pre-published.

[Sat] Christian Sattler. “CYLINDRICAL MODEL STRUCTURES.” In: (). URL: https://www.cse.chalmers.se/~sattler/docs/interval-model-structure.pdf.

[Sat17] Christian Sattler. *The Equivalence Extension Property and Model Structures*. Apr. 23, 2017. URL: https://arxiv.org/abs/1704.06911v4 (visited on 08/23/2023). Pre-published.

[Sat19] Christian Sattler. *Idempotent Completion of Cubes in Posets*. Mar. 8, 2019. DOI: 10.48550/arXiv.1805.04126. arXiv: 1805.04126 [math]. URL: http://arxiv.org/abs/1805.04126 (visited on 08/23/2023). Pre-published.

[Shu] Michael Shulman. “REEDY CATEGORIES AND THEIR GENERALIZATIONS.” In: ().

[Sta] The {Stacks project authors}. *The Stacks Project*. In: URL: https://stacks.math.columbia.edu/tag/00VC.

[SW21] Thomas Streicher and Jonathan Weinberger. “Simplicial Sets inside Cubical Sets.” In: *Theory and Applications of Categories* 37.10 (Mar. 15, 2021), pp. 276–286. URL: http://www.tac.mta.ca/tac/volumes/37/10/37-10abs.html (visited on 05/02/2024).